DE PRASSE
THEORIA
AEQUATIONUM
N. 22 T. 1

VITTORIO EM. III

...DO PIZZOFALCONE

NAZIONALE
BIBLIOTECA
B. Prov.
Miscellanea
C
43
296
VITTORIO EM. III
NAPOLI

BIBLIOTECA PROVINCIALE
mis. C. 43-296
Armadio XXVII
Palchetto
Num.° d' ordine 62
26700

Digitized by Google

Digitized by Google

USUS

LOGARITHMORUM INFINITINOMII

IN

THEORIA AEQUATIONUM

AUCTORE

MAURICIO DE PRASSE

BIBLIOTECA NAZIONALE VITTORIO EMANUELE III
NAPOLI

ADIECTA EST TABULA SINGULARIS

LIPSIAE
APUD CHRIST. THEOPH. RABENHORST
1796.

Digitized by Google

Digitized by Google

§. I.

Institvti Ratio.

Continentur hoc libello Analyseos Combinatoriae *) quaeſtiones tres ex plurimis aliis, quae forma aeque univerſali, ut hic apparent, nusquam, quod equidem ſciam, pertractatae reperiuntur, et quarum inveſtigandarum occaſionem praebuit ***mutua relatio exponentium et coefficientium in aequatione:***

$$[1-(ax+bx^2..)]^{\alpha}[1-(\mathfrak{a}x+\mathfrak{b}x^2..)]^{\beta}\ldots[1-(\mathfrak{A}x+\mathfrak{B}x^2..)]^{f}=1-(\mathrm{A}x+\mathrm{B}x^2+\mathrm{C}x^3\ldots+\mathrm{N}x^n..)$$

Propter *Methodi Combinatoriae* in quaeſtionum analyticarum ſolutione hic adhibitae ipſarumque quaeſtionum indolem viſum eſt mihi, non abs re eſſe, propoſitiones aliquot nondum ſatis notas praeparationis loco praemittere, quibus *formulae dignitatum Polynomii* diverſae et quaedam de *Combinatoriis Involutionibus* hoc in libello obviis, continentur, ut deinde §. XXVII ſeqq. remotis omnibus impedimentis, rei poſſit ſumma afferri ejusque uſus exemplis nonnullis illuſtrari.

Si quid ego effecerim, combinatoriis involutionibus ad Analyſeos theoremata ſummi momenti translatis, eam ob rem ſtudium, quo me disciplinis Matheſeos tradendis dicavi, cuique ut probetur ab illis non alieno, jam in votis habeo: Peritis vero ſatisfacere non, niſi in poſterum, conabor.

§. II.

*) Theoriam artis combinatoriae ejusque ad Analyſin applicationem, Hindenburgius, *Vir Celeberrimus*, in opere, quod inſcribitur: *Novi Syſtematis Permutationum, Combinationum ac Variationum primae lineae* (Lipſiae 1781.), primus ita propoſuit, ut in ea, veluti in fundamento, calculus nitatur omnis. Multa etiam continent ad rem combinatoriam et Analyſin pertinentia, propoſitis ſimul plurimis et exemplis et tabulis combinatoriis, ipſius: *Infinitinomii Dignitatum Hiſtoria, Leges ac Formulae.* . . (Gottingae 1779.). Quibus vero meditationibus peculiaribus, qua via quibusque artificiis ad memorabile illud inventum perductus fuerit, docuit nuper, omnia ſimul enumerans ſcripta eo pertinentia: *Archiv der reinen und angewandten Mathematik.* (Leipzig bey Schäfer 1794.) Heft II. pag. 242.

Digitized by Google

§. II.

Definitiones.

1) Ars combinatoria res datas ex ordine notat ſive literis *a, b, c, d, e, f, g, h.* . . . . ; ſive numeris 1, 2, 3, 4, 5, 6, 7, 8 . . . . . ; Signa ſingula *Elementorum,* collecta vero $\left(\begin{matrix} a, & b, & c, & d, & e, & f & \ldots \\ 1, & 2, & 3, & 4, & 5, & 6 & \ldots \end{matrix}\right)$ *Indicis* nomen habent.

2) Elementorum conjunctiones Singulae, veluti *abc*; *ddd*; *aecbd*; ſive 123; 444; 15324; *Complexiones* vocantur, et quidem *rite ordinatae*, in quibus (ſi a ſiniſtra ad dextram legas) elementum poſterius nullum antepoſitum eſt priori, e. g. *aab*; *abc*; *abcd*; 112; 123; 1234; Itaque *aba*; *cab*; *abſee*; *aaacb*; 121; 312; 12655; 11132; non ſunt rite ordinatae.

3) Complexiones dividuntur

in *Biniones*, quales ſunt *aa*; *ab*; *bc*; 11; 12; 23;
" *Terniones*, " " *aaa*; *abc*; *cbd*; 111; 123; 324;
" *Quaterniones* " " *abbc*; *adef*; 1223; 1456;
" " " " " " " " "
" mtiones " " *abcd* .... *m*; 1234....m;

prout binis, ternis, quaternis, ... m Elementis conſtant. Ad hujus appellationis analogiam ſingula elementa ipſa dicuntur *Vniones* v. c. *a*; *b*; *c*; 1; 2; 3;

4) Cum reſpicitur ad aliquam conjungendi legem, ſecundum quam ipſae complexiones producuntur,

collectio Vnionum vocatur *prima Claſſis*
" Binionum " *ſecunda* "
" Ternionum " *tertia* "
" " " " "
" mtionum " *mta* "

5) *Claſſis rite ordinata* dicitur, cujus complexiones omnes ad inſtar numerorum creſcentium procedunt.

Sic v. c. Complexiones $\left(\begin{matrix} 116; & 125; & 134; & 224; & 233 \\ aaf; & abe; & acd; & bbd; & bcc \end{matrix}\right)$; non autem $\left(\begin{matrix} 224; & 134; & 233; & 116; & 125 \\ bbd; & acd; & bcc; & aaf; & abe \end{matrix}\right)$ aut aliter diſpoſitae, ut numeri creſcentes, progrediuntur.

6) *Discerptiones numeri n* eae vocantur complexiones numericae, in quibus *elementorum* cujusque complexionis *ſumma* *) aequat numerum *n*.

7) Com-

*) In complexione *ſumma* elementorum cum ipſorum *numero* confundendus non eſt. Illa ab ipſorum elementorum numericorum *magnitudine*, hic a multitudine pendet, et in eadem claſſe conſtanter

7) *Combinationes ſummae propoſitae n* eae dicuntur complexiones numericae rite ordinatae (Def. 2), quae ſimul diſcerptiones ſunt numeri *n* (Def. 6); quo pertinent v. c. ſummae 5 combinationes a latere collocatae et per claſſes (Def. 4.) diſpoſitae.

5

14
23

113
122

1112

11111

8) *Permutari* datae combinationis elementa dicuntur, ubi omnibus, quibus poſſunt, modis ſedibus transponuntur ſuis, et prodeuntes permutando conjunctiones ipſae *Permutationes datae combinationis* vocantur.

§. III.

PROBLEMA.

Data Combinatione Summae n (§. II. Def. 7.) *reperire proxime ſequentem Claſſis rite ordinatae* (§. II. Def. 5) et *indicis* (1, 2, 3, 4 ....) (§. Def. 1).

Combinatio data ſit v. c. 112445; ubi $n = 17$ et index (1, 2, 3, 4, 5, 6 ....)

SOLVTIO.

I. In data Combinatione quaeratur, a dextra ad ſiniſtram eundo, numerus primus, qui *duabus ſaltem unitatibus* differat a numero extremo ad dextram et *inventus* (hic 2) *unitate augeatur*. Si talis numerus non reperitur, combinatio data ipſa eſt claſſis ſuae ultima, veluti 233333.

II. Numerus ſecundum I. auctus (jam 3) in omnibus ad dextram ſedibus (quae adſunt) collocetur, excepta tamen extrema.

III. Numeri ad ſiniſtram (ſi qui adſunt) maneant immutati.

IV. In ſede ad dextram extrema ponatur complementum ſummae *n*. (h. l. complementum eſt 6)

Vnde

ſtanter idem eſt. Sic in complexione 13158 *numerus* elementorum eſt 5, propter quinque Elementa 1; 3; 1; 5; 8; Elementorum vero *ſumma* eſt $1+3+1+5+8 = 18$. In complexione 25412 numerus elementorum iterum eſt 5, ſumma autem 14.

Vnde ſit:

		ex	112445
Secundum	I.		..3...
ſ	I. II.		..333.
ſ	I. II. III.		11333.
ſ	I. II. III. IV.		113336

		ex	113336
Secundum	I.		....4.
ſ	I. II.		....4.
ſ	I. II. III.		11334.
ſ	I. II. III. IV.		113345

		ex	113345
Secundum	I.		...4..
ſ	I. II.		...44.
ſ	I. II. III.		113444
ſ	I. II. III. IV.		113444

Demonstratio.

Patet, illis regulis Diſcerptionem reperiri. Diſcerptio vero rite ordinata eſt, quod numerum auctum ſequuntur numeri nulli ipſo minores (reg. III. et IV.). Combinationem denique prodire datae proximam, intelligitur eo, quod ultimam ſedem tenet numerus maximus eorum, quos, ubi regulae I. ſatisfactum eſt, poni licuiſſet.

§. IV.

Problema.

Combinationum ſummae propoſitae n claſſem quamlibet, v. c. mtam (vbi $m < n$) conſtruere, dato Indice (1, 2, 3, 4 . . . .).

Solutio.

1) Scribantur $(m - 1)$ unitates, alia juxta aliam, et vltimo loco complementum ad Summam n; i. e. $(n - m + 1)$.

2) Ex prima hac combinatione per regulas §. III. deducatur ſecunda, ex ſecunda tertia atque ita quaelibet poſterior ex proxime priori, donec iſtae regulae amplius adhiberi nequeant, atque omnes Combinationes expreſſae erunt.

Exemplum. Si $n = 10$, $m = 5$, producitur

ſecundum 1) prima combinatio: 1 1 1 1 6

ſ 2) ſucceſſive reliquae omnes:

1 1 1 2 5
1 1 1 3 4
1 1 2 2 4
1 1 2 3 3
1 2 2 2 3
2 2 2 2 2 *).

§. V.

*) De numerorum diſcerptionibus egerunt Hindenburgius, Vir Celeberrimus, (*Infinit. Dignitatum Hiſt. Leg. ac Form.* p. 73. ſeqq. et p. 129. ſeqq. et *Progr.* quo *Terminorum ab infinitinomii dignitatibus Coefficientes* Moivraeanos *ſequi ordinem lexicographicum oſtenditur* Lipſiae 1795.) et Toepferus, Vir Clariſſimus, (*Combinatoriſche Analytik* etc. p. 68.)

§. V.

Explicatio.

Combinationum ſummae propoſitae n claſſes (§. II. Def. 7.) ex ordine per literas majores Latinas A, B, C, D, ... et numerum n, ſequentem in modum, notantur: Sit v. c. ſumma propoſita $n = 8$, notabitur

prima claſſis combinationum ſummae 8	i. e.	8	Signo ^{8}A
ſecunda	. . . . .	17, 26, 35, 44	. ^{8}B
tertia	. . . . .	116, 125, 134, 224, 233	. ^{8}C
quarta	. . . . .	1115, 1124, 1133, 1223, 2222	. ^{8}D
quinta	. . . . .	11114, 11123, 11222	. ^{8}E
ſexta	. . . . .	111113, 111122	. ^{8}F
ſeptima	. . . . .	1111112	. ^{8}G
octava	. . . . .	11111111	. ^{8}H

Index eſt $_{(1,2,3,4,5,6,7,8)}$

Numerus n (hic 8), qui literae majori hic iungitur a laeva, dicitur *Exponens ſummae* (der Summenexponent). Literae cum numeris *Signa Claſſium* conſtituunt.

Scholion I. Index ſignis combinatoriis ſemper adiiciendus, quod, illo omiſſo, combinatoria ſigna ipſa intelligi nequeunt.

Scholion II. Signa $^{n}\underset{(1,2,3\ldots n)}{M}$ et $^{n}\underset{(1,2,3\ldots n)}{\mathcal{M}}$ diligenter distinguenda ſunt. Illud enim *claſſem* combinationum ſummae *n duodecimam*, quia M duodecima eſt litera alphabeti A, B, C, D, . . ., hoc vero, in quo $\mathcal{M}$ alterius eſt alphabeti, *mtam* generaliter *claſſem exprimit*.

§. VI.

Digitized by Google

§. VI.

Definitio.

Combinationes ſummae propoſitae n ex elementis literalibus eae vocantur Complexiones, quae ex numericis ejusdem ſummae, ſecundum problema §. IV. productis, oriuntur, ſi loco numerorum (1, 2, 3, 4 ....) literae his reſpondentes ponantur, hoc modo:

$6 = {}^{6}A$	$f = {}^{6}A$	$g = {}^{6}A$
$\left.\begin{matrix}15\\24\\33\end{matrix}\right\} = {}^{6}B$	$\left.\begin{matrix}ae\\bd\\cc\end{matrix}\right\} = {}^{6}B$	$\left.\begin{matrix}bf\\ce\\dd\end{matrix}\right\} = {}^{6}B$
$\left.\begin{matrix}114\\123\\222\end{matrix}\right\} = {}^{6}C$	$\left.\begin{matrix}aad\\abc\\bbb\end{matrix}\right\} = {}^{6}C$	$\left.\begin{matrix}bbe\\bcd\\ccc\end{matrix}\right\} = {}^{6}C$
$\left.\begin{matrix}1113\\1122\end{matrix}\right\} = {}^{6}D$	$\left.\begin{matrix}aaac\\aabb\end{matrix}\right\} = {}^{6}D$	$\left.\begin{matrix}bbbd\\bbcc\end{matrix}\right\} = {}^{6}D$
$11112 = {}^{6}E$	$aaaab = {}^{6}E$	$bbbbc = {}^{6}E$
$111111 = {}^{6}F$	$aaaaaa = {}^{6}F$	$bbbbbb = {}^{6}F$
(1, 2, 3, 4, 5, 6)	$\binom{a, b, c, d, e, f}{1, 2, 3, 4, 5, 6}$	$\binom{b, c, d, e, f, g}{1, 2, 3, 4, 5, 6}$

ubi indices inferius hic appoſiti oſtendunt, ad quos numeros literae pertinent. Hinc intelligitur, quid futurum eſſet, ſi indices fuiſſent

$$\binom{c, d, e, f, g, h}{1, 2, 3, 4, 5, 6} \text{ vel } \binom{d, e, f, g, h, i}{1, 2, 3, 4, 5, 6}$$

§. VII.

Explicatio.

Praefixae ſignis claſſium (§. V.) homonymae literae Germanicae minores, veluti

$$\mathfrak{a}^{n}A;\ \mathfrak{b}^{n}B;\ \mathfrak{c}^{n}C;\ \mathfrak{d}^{n}D;\ \mathfrak{e}^{n}E;\ \ldots\ldots\ldots\ \mathfrak{m}^{n}\mathcal{M}$$

$$\binom{b,\ c,\ d,\ e,\ f,\ g\ \ldots\ldots}{1,\ 2,\ 3,\ 4,\ 5,\ 6\ \ldots\ldots}$$

indicant, ſingulis Complexionibus literalibus praeponendum eſſe *Numerum permutationum* *), quo ſcilicet docetur, quoties complexionis elementa ſuis transponi queant ſedibus (Def. 8.). Sic, poſito $n = 8$, eſt:

$\mathfrak{t}^{8}D =$

*) Numerus permutatiorum complexionis $b\ c\ d\ e\ f\ g\ h \ldots\ldots$; ubi elementa numero m ſunt omnia diverſa, eſt $m.\, m - 1.\, m - 2 \ldots\ldots 3.\, 2.\, 1.$ Quod ſi autem inter haec aliqua ſunt eadem numero β et alia numero γ et alia numero δ etc. fit numerus permutationum

=

Digitized by Google

$$\underset{\left(\begin{smallmatrix} b, & c, & d, & e, & f \ldots \\ 1, & 2, & 3, & 4, & 5 \ldots \end{smallmatrix}\right)}{\mathfrak{t}^8 D} = \left\{\begin{matrix} 4bbbf \\ 12bbce \\ 6bbdd \\ 12bccd \\ 1cccc \end{matrix}\right\} \quad \text{et} \quad \underset{\left(\begin{smallmatrix} b, & c, & d, & e \ldots \\ 1, & 2, & 3, & 4 \ldots \end{smallmatrix}\right)}{\mathfrak{e}^8 E} = \left\{\begin{matrix} 5bbbbe \\ 20bbbcd \\ 10bbccc \end{matrix}\right\}$$

§. VIII.

Theorema.

Si $y = bx + cx^2 + dx^3 + ex^4 + \ldots\ldots\ldots$, potentiae integrae poſitivae hujus ſeriei, ſignis §. VII. explicatis expreſſae, ex ordine ſunt:

$$y^1 = \mathfrak{a}^1 A x + \mathfrak{a}^2 A x^2 + \mathfrak{a}^3 A x^3 + \mathfrak{a}^4 A x^4 + \mathfrak{a}^5 A x^5 + \mathfrak{a}^6 A x^6 + \ldots\ldots + \mathfrak{a}^n A x^n + \ldots$$
$$y^2 = \mathfrak{b}^2 B x^2 + \mathfrak{b}^3 B x^3 + \mathfrak{b}^4 B x^4 + \mathfrak{b}^5 B x^5 + \mathfrak{b}^6 B x^6 + \ldots\ldots + \mathfrak{b}^n B x^n + \mathfrak{b}^{n+1} B x^{n+1} + \ldots$$
$$y^3 = \mathfrak{c}^3 C x^3 + \mathfrak{c}^4 C x^4 + \mathfrak{c}^5 C x^5 + \mathfrak{c}^6 C x^6 + \ldots\ldots + \mathfrak{c}^n C x^n + \mathfrak{c}^{n+1} C x^{n+1} + \mathfrak{c}^{n+2} C x^{n+2} + \ldots$$
$$\text{〃} \quad \text{〃} \quad \text{〃} \quad \text{〃} \quad \text{〃} \quad \text{〃} \quad \text{〃} \quad \text{〃} \quad \text{〃}$$
$$y^m = \mathfrak{m}^m \mathfrak{M} x^m + \mathfrak{m}^{m+1} \mathfrak{M} x^{m+1} + \mathfrak{m}^{m+2} \mathfrak{M} x^{m+2} + \ldots + \mathfrak{m}^n \mathfrak{M} x^n \ldots\ldots + \mathfrak{m}^{n+m-1} \mathfrak{M} x^{n+m-1} \ldots$$
$$\left(\begin{matrix} b, & c, & d, & e, & f, & g & \ldots\ldots \\ 1, & 2, & 3, & 4, & 5, & 6 & \ldots\ldots \end{matrix}\right)$$

Termini hic ſimul apparent generales dignitatum y^1, y^2, y^3, ... y^m.

Hujus Theorematis demonſtratio extat in *Infinit. Dign.* §. XXIII. Cf. *Nov. Syſt.* p. LIV. 8.

§. IX.

Problema.

Dignitatis y^m §. VIII. terminum conſtruere quemlibet independenter v. c. poſito $m = 4$, dignitatis y^4 terminum ſeptimum.

Solu-

$= \frac{m.\,m-1.\,m-2.\,m-3\,\ldots\ldots\ldots\,3.\,2.\,1.}{1.\,2.\,3\,\cdot\,\beta.\,1.\,2\ldots\,\gamma.\,1.\,2.\,3\ldots\,\delta\ \text{etc.}}$ Hine numerus permutationum combinationis *b c d e f g* eſt 6. 5. 4. 3. 2. 1 $= 720$ complexionis autem $b^2 c^3 d = \frac{6.\,5.\,4.\,3.\,2.\,1}{1.\,2.\,1.\,2.\,3.\,1} = 60$. (Cf. Hindenburgii *Infinit. Dign.* §. XIII. p. 32. *Nov. Syſt. Perm.* p. 24. *KAESTNERI, VIRI SVMMI, Anal. endl. Größen* ed. 1794. §. 34.) Hunc numerum (combinatorium ſcilicet origine), in complexionibus formularum polynomii occurrentem, *coefficientem polynomialem* dixit Hindenburgius (*Nov. Syſt.* p. IX.) ad analogiam Coefficientis binomialis, quocum etiam ſaepiſſime conjungitur.

Digitized by Google

S o l v t i o.

Ex termino *n*to $\mathfrak{m}^{n+m-1}\mathfrak{M}\, x^{n+m-1}$ $\binom{b,\, c,\, d \ldots}{1,\, 2,\, 3 \ldots}$ ſeriei y^m (§. VIII.) ſequitur, poſito $n = 7$ et $m = 4$, dignitatis y^4 terminus ſeptimus $\mathfrak{b}^{10}\mathfrak{D}x^{10}$ $\binom{b,\, c,\, d \ldots}{1,\, 2,\, 3 \ldots}$. Itaque, ut producatur terminus quaeſitus,

I. Diſcerptionum numeri 10. claſſis conſtruatur quarta ſecundum §. IV.	II. Numerorum loco ponantur literae reſpondentes (§. VI.) ſecundum Indicem $\binom{b, c, d, e, f, g \ldots}{1, 2, 3, 4, 5, 6 \ldots}$	III. Complexionibus literalibus praefigantur numeri permutationum* (Cf. §. VII. *) et ſingulis poteſtas x^{10} jungatur.
1117	*bbbh*	4*bbbh* x^{10}
1126	*bbcg*	12*bbcg*
1135	*bbdf*	12*bbdf*
1144	*bbee*	6*bbee*
1225	*bccf*	12*bccf*
1234	*bcde*	24*bcde*
1333	*bddd*	4*bddd*
2224	*ccce*	4*ccce*
2233	*ccdd*	6*ccdd*

Eodem modo terminus quilibet poteſtatis y^4 in ſchemate ſequenti reperitur:

	(4)	(5)	(6)	(7)	(8)	(9)
Fit ſecundum I.	1111	1112	1113	1114	1115	1116
			1122	1123	1124	1125
				1222	1133	1134
					1223	1224
					2222	1233
						2223
ſecundum II.	*bbbb*	*bbbc*	*bbbd*	*bbbe*	*bbbf*	*bbbg*
			bbcc	*bbcd*	*bbce*	*bbcf*
				bccc	*bbdd*	*bbde*
					bccd	*bcce*
					cccc	*bcdd*
						cccd

Secun-

Digitized by Google

ſecundum III. $y^4 = 1bbbb\,x^4 + 4bbbc\,x^5 + \left.\begin{array}{r}4bbbd\\ 6bbcc\end{array}\right| x^6 + \left.\begin{array}{r}4bbbe\\ 12bbcd\\ 4bccc\end{array}\right| x^7 + \left.\begin{array}{r}4bbbf\\ 12bbce\\ 6bbdd\\ 12bccd\\ 1cccc\end{array}\right| x^8 + \left.\begin{array}{r}4bbbg\\ 12bbcf\\ 12bbde\\ 12bcce\\ 12bcdd\\ 4cccd\end{array}\right| x^9 + \ldots$

Scholion. Terminos quaeſitos integros dignitatis y^4 ſeorſim h. l. ſcripſimus idque fecimus majoris perſpicuitatis gratia, ſed non opus eſt hac diſtinctione, quia ſtatim poteſt terminus quilibet per complexiones in II. exhiberi, addendo numeros permutationum complexionibus ſingulis a laeva, variabilis autem x poteſtatem reſpondentem a dextra. Exercitatus paululum poterit etiam, praetereundo I. illico ſcribere complexiones literales in II. et inde conficere ***terminum quaeſitum*** (III.), qua re totum negotium et contrahitur et ſublevatur.

§. X.

Explicatio.

Coefficientes binomiales, cujuslibet exponentis v, ſequentem in modum notantur:

$^{v}\mathfrak{A}$ exprimit $\frac{v}{1}$

$^{v}\mathfrak{B}$ „ $\frac{v\,.\,(v-1)}{1\,.\,2}$

$^{v}\mathfrak{C}$ „ $\frac{v\,.\,(v-1)\,.\,(v-2)}{1\,.\,2\,.\,3}$

$^{v}\mathfrak{D}$ „ $\frac{v\,.\,(v-1)\,.\,(v-2)\,.\,(v-3)}{1\,.\,2\,.\,3\,.\,4}$

„ „ „ „

$^{v}\mathfrak{M}$ „ $\frac{v\,.\,(v-1)\,.\,(v-2)\,\ldots\,(v-m+1)}{1\,.\,2\,.\,3\,\ldots\ldots\,m}$

Scholion I. Liquet m nullum hic niſi integrum poſitivum ſignificare poſſe numerum. Coefficiens binomialis *generalis*, hoc loco $\mathfrak{M}$us, litera alterius alphabeti notatur, quo cognoſcatur, eum eſſe generalem.

Scholion II. Jam ſigna quoque ſequentia ex ſignis §. X et VII. compoſita facile intelligentur v. c.

$$^{6}\mathfrak{C}\,c\,^{8}C\left(\begin{array}{c}b, c, d, e, \ldots\\ 1, 2, 3, 4, \ldots\end{array}\right)$$

quod jubet, conſtrui tertiam Claſſem combinationum ſummae 8 ex elementis $(b, c, d, e \ldots)$, quae eſt

$$^{8}C\left(\begin{array}{c}b, c, d, e, \ldots\\ 1, 2, 3, 4, \ldots\end{array}\right) = \left\{\begin{array}{l}b^2g\\ bcf\\ bde\\ c^2e\\ cd^2\end{array}\right.$$

dein numerum permutationum cuivis complexioni praeponi, unde fit

Digitized by Google

$$\underset{\binom{b,\,c,\,d,\,e\,\ldots}{1,\,2,\,3,\,4\,\ldots}}{c^8C} = \left[\begin{array}{l} 3b^2g \\ 6bcf \\ 6bde \\ 3c^2e \\ 3cd^2 \end{array}\right]$$

complexionum denique ſummam multiplicari per productum ${}^6\mathfrak{C} = \frac{6.\,5.\,4}{1.\,2.\,3}$; quo producitur:

$$\underset{\binom{b,\,c,\,d,\,e\,\ldots}{1,\,2,\,3,\,4\,\ldots}}{{}^6\mathfrak{C}\,c^8C} = \frac{6.\,5.\,4}{1.\,2.\,3}\left[\begin{array}{l} +\ 3b^2g \\ +\ 6bcf \\ +\ 6bde \\ +\ 3c^2e \\ +\ 3cd^2 \end{array}\right]$$

§. XI.

Theorema.

Sit $az^{\mu} + bz^{\mu+\delta} + cz^{\mu+2\delta}\, dz^{\mu+3\delta} + \ldots = \mathrm{p}.$

Erit $$\mathrm{p}^{\nu} = a^{\nu}z^{\nu\mu} + {}^{\nu}\mathfrak{A}a^{\nu-1}\mathfrak{a}^1\mathrm{A}z^{\nu\mu+\delta} + \left.\begin{array}{l} {}^{\nu}\mathfrak{A}a^{\nu-1}\mathfrak{a}^2\mathrm{A} \\ +\,{}^{\nu}\mathfrak{B}a^{\nu-2}\mathfrak{b}^1\mathrm{B} \end{array}\right| z^{\nu\mu+2\delta} + \left.\begin{array}{l} {}^{\nu}\mathfrak{A}a^{\nu-1}\mathfrak{a}^3\mathrm{A} \\ +\,{}^{\nu}\mathfrak{B}a^{\nu-2}\mathfrak{b}^1\mathrm{B} \\ +\,{}^{\nu}\mathfrak{C}a^{\nu-3}\mathfrak{c}^1\mathrm{C} \end{array}\right| z^{\nu\mu+3\delta} \ldots + \left.\begin{array}{llll} {}^{\nu}\mathfrak{A} & a^{\nu-1} & \mathfrak{a} & {}^{n}\mathrm{A} \\ +\,{}^{\nu}\mathfrak{B} & a^{\nu-2} & \mathfrak{b} & {}^{n}\mathrm{B} \\ +\,{}^{\nu}\mathfrak{C} & a^{\nu-3} & \mathfrak{c} & {}^{n}\mathrm{C} \\ & \vdots & & \\ +\,{}^{\nu}\mathfrak{M} & a^{\nu-m} & \mathfrak{m} & {}^{n}\mathcal{M} \\ & \vdots & & \\ +\,{}^{\nu}\mathfrak{N} & a^{\nu-n} & \mathfrak{n} & {}^{n}\mathcal{N} \end{array}\right| z^{\nu\mu+n\delta} \ldots$$

$$\binom{b,\ c,\ d,\ e,\ f,\ \ldots\ldots}{1,\ 2,\ 3,\ 4,\ 5,\ \ldots\ldots}$$

ubi μ, δ *et* ν *numeri quicunque poſitivi, negativi, integri, fracti eſſe poſſunt.*

Demonstratio.

Adhibitis ſignis coefficientium binomialium (§. X.), fit binomium

$(a + y)^{\nu} = a^{\nu} + {}^{\nu}\mathfrak{A}a^{\nu-1}y^1 + {}^{\nu}\mathfrak{B}a^{\nu-2}y^2 + {}^{\nu}\mathfrak{C}a^{\nu-3}y^3 + {}^{\nu}\mathfrak{D}a^{\nu-4}y^4 + \ldots\ldots$

Jam, ſi ponitur

$y = bx + cx^2 + dx^3 + ex^4 + fx^5 + \ldots\ldots$

facta ſubſtitutione potentiarum hujus ſeriei, §. VIII. exhibitarum, ſequitur:

$(a + bx$

Digitized by Google

$$
(a+bx+cx^2+dx^3+\ldots)^r = \left.\begin{array}{l}
a^r + {}^r\mathfrak{A}\,a^{r-1}\,y^1 \\
+ {}^r\mathfrak{B}\,a^{r-2}\,y^2 \\
+ {}^r\mathfrak{C}\,a^{r-3}\,y^3 \\
\quad\vdots \\
+ {}^r\mathfrak{M}\,a^{r-m}\,y^m \\
\quad\vdots \\
+ {}^r\mathfrak{N}\,a^{r-n}\,y^n \\
\quad\vdots
\end{array}\right| = \begin{array}{l}
a^r + {}^r\mathfrak{A}\,a^{r-1}\,[\mathfrak{a}^1Ax + \mathfrak{a}^2Ax^2 + \mathfrak{a}^3Ax^3 + \ldots + \mathfrak{a}^nAx^n + \ldots] \\
+ {}^r\mathfrak{B}\,a^{r-2}\,[\mathfrak{b}^2Bx^2 + \mathfrak{b}^3Bx^3 + \ldots + \mathfrak{b}^nBx^n + \ldots] \\
+ {}^r\mathfrak{C}\,a^{r-3}\,[\mathfrak{c}^3Cx^3 + \ldots + \mathfrak{c}^nCx^n + \ldots] \\
\quad\vdots \\
+ {}^r\mathfrak{M}\,a^{r-m}\,[\mathfrak{m}^m\mathcal{M}x^m + \ldots + \mathfrak{m}^n\mathcal{M}x^n \ldots] \\
\quad\vdots \\
+ {}^r\mathfrak{N}\,a^{r-n}\,[\mathfrak{n}^n\mathcal{N}x^n + \ldots] \\
\quad\vdots
\end{array}
\begin{pmatrix} b, c, d, e, f, \ldots \\ 1, 2, 3, 4, 5, \ldots \end{pmatrix}
$$

terminisque ſecundum potentias quantitatis x digeſtis:

$$
(a+bx+cx^2+dx^3+\ldots)^r =
$$

$$
a^r + {}^r\mathfrak{A}a^{r-1}\mathfrak{a}^1Ax^1 + \left.\begin{array}{l} {}^r\mathfrak{A}a^{r-1}\mathfrak{a}^2A \\ + {}^r\mathfrak{B}a^{r-2}\mathfrak{b}^2B \end{array}\right| x^2 + \left.\begin{array}{l} {}^r\mathfrak{A}a^{r-1}\mathfrak{a}^3A \\ + {}^r\mathfrak{B}a^{r-2}\mathfrak{b}^3B \\ + {}^r\mathfrak{C}a^{r-3}\mathfrak{c}^3C \end{array}\right| x^3 + \ldots\ldots + \left.\begin{array}{l} {}^r\mathfrak{A}\,a^{r-1}\,\mathfrak{a}^nA \\ + {}^r\mathfrak{B}\,a^{r-2}\,\mathfrak{b}^nB \\ + {}^r\mathfrak{C}\,a^{r-3}\,\mathfrak{c}^nC \\ \quad\vdots \\ + {}^r\mathfrak{M}a^{r-m}\mathfrak{m}^n\mathcal{M} \\ \quad\vdots \\ + {}^r\mathfrak{N}a^{r-n}\,\mathfrak{n}^n\mathcal{N} \end{array}\right| x^n + \ldots
$$

$$
\begin{pmatrix} b, & c, & d, & e, & f, \ldots \\ 1, & 2, & 3, & 4, & 5, \ldots \end{pmatrix}
$$

vnde formula prodit Theorematis, poſito utrinque $x = z^3$, additoque factore $z^{r\mu}$. (Hindenburgius *Nov. Syſt.* p. LIV. 7.)

Scholion. Formulae dignitatum Infinitinomii, hic et §. VIII. propoſitae, vocantur *combinatoriae*, quod ſignis utuntur combinatoriis. Pluribus terminis formula theorematis exhibetur in tabula adjecta.

§. XII.

Problema.

Dignitatis p^r (§.XI.) *conſtruere terminum quemlibet independenter, v. e. undecimum.*

Solutio.

SOLUTIO.

10	$+{}^{\nu}\mathfrak{A}a^{\nu-1}$	$[\,1l\,]$	$={}^{\nu}\mathfrak{A}a^{\nu-1}\mathfrak{a}^{10}A$ $z^{\nu\mu+10\delta}$
19	$+{}^{\nu}\mathfrak{B}a^{\nu-2}$	$2bk$	$={}^{\nu}\mathfrak{B}a^{\nu-2}\mathfrak{b}^{10}B$
28		$2ci$	
37		$2dh$	
46		$2eg$	
55		$1ff$	
118	$+{}^{\nu}\mathfrak{C}a^{\nu-3}$	$3bbi$	$={}^{\nu}\mathfrak{C}a^{\nu-3}\mathfrak{c}^{10}C$
127		$6bch$	
136		$6bdg$	
145		$6bef$	
226		$3ccg$	
235		$6cdf$	
244		$3cee$	
334		$3dde$	
1117	$+{}^{\nu}\mathfrak{D}a^{\nu-4}$	$4bbbh$	$={}^{\nu}\mathfrak{D}a^{\nu-4}\mathfrak{d}^{10}D$
1126		$12bbcg$	
1135		$12bbdf$	
1144		$6bbee$	
1225		$12bccf$	
1234		$24bcde$	
1333		$4bddd$	
2224		$4ccce$	
2233		$6ccdd$	
11116	$+{}^{\nu}\mathfrak{E}a^{\nu-5}$	$5bbbbg$	$={}^{\nu}\mathfrak{E}a^{\nu-5}\mathfrak{e}^{10}E$
11125		$20bbbcf$	
11134		$20bbbde$	
11224		$30bbcce$	
11233		$30bbcdd$	
12223		$20bcccd$	
22222		$1ccccc$	
111115	$+{}^{\nu}\mathfrak{F}a^{\nu-6}$	$6bbbbbf$	$={}^{\nu}\mathfrak{F}a^{\nu-6}\mathfrak{f}^{10}F$
111124		$30bbbbce$	
111133		$15bbbbdd$	
111223		$60bbbccd$	
112222		$15bbcccc$	
1111114	$+{}^{\nu}\mathfrak{G}a_{\nu-7}$	$7bbbbbbe$	$={}^{\nu}\mathfrak{G}a^{\nu-7}\mathfrak{g}^{10}G$
1111123		$42bbbbbcd$	
1111222		$35bbbbccc$	
11111113	$+{}^{\nu}\mathfrak{H}a^{\nu-8}$	$8bbbbbbbd$	$={}^{\nu}\mathfrak{H}a^{\nu-8}\mathfrak{h}^{10}H$
11111122		$28bbbbbbcc$	
111111112	$+{}^{\nu}\mathfrak{I}a^{\nu-9}$	$[\,9bbbbbbbbc\,]$	$={}^{\nu}\mathfrak{I}a^{\nu-9}\mathfrak{i}^{10}I$
1111111111	$+{}^{\nu}\mathfrak{K}a^{\nu-10}$	$[\,1bbbbbbbbbb\,]$	$={}^{\nu}\mathfrak{K}a^{\nu-10}\mathfrak{k}^{10}K$

1) Numerus termini quaeſiti (h. l. 11) unitate minuatur, diſcerptionumque numeri reſidui claſſes ex ordine omnes conſtruantur (§. IV.)

2) Loco numerorum ſubſtituantur (§. VI.) literae reſpondentes, ſecundum indicem $\binom{b, c, d, e, f, g, h, i, k, l, \ldots}{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots}$

3) Complexionibus ſingulis praefigantur numeri permutationum literarum (§. VII.)

4) ${}^{\nu}\mathfrak{A}a^{\nu-1}$ praeponatur *primae* claſſi
${}^{\nu}\mathfrak{B}a^{\nu-2}$ . . *ſecundae* .
${}^{\nu}\mathfrak{C}a^{\nu-3}$. *tertiae* .
${}^{\nu}\mathfrak{D}a^{\nu-4}$. *quartae* .
. .
. .
(vide §. X. Scholion *II*)

5) Complexionibus omnibus addatur factor communis $z^{\nu\mu+10\delta}$.

DEMONSTRATIO.

Haec terminorum conſtructio per partes congruit cum termino generali formulae dignitatum infinitinomii combinatoriae §. XI. Q. E. F.

Digitized by Google

Scholion. Termini ſi omnes ſeriei p^ν evolvuntur eodem modo, quo hic undecimus, prodit *formula dignitatum infinitinomii algebraice expreſſa.* Hujus termini primi decem in tabula adjecta conſpiciuntur, eamque ob cauſam hoc loco undecimus evolutus eſt.

§. XIII.

Explicatio.

Sit $p = az^\mu + bz^{\mu+\beta} + cz^{\mu+2\beta} + \ldots$ uti §. XI, *Coefficientes* dignitatis p^ν ex ordine his etiam notantur ſignis:

$p^\nu\varkappa 1$ exprimit *coefficientem primum* i.e. a^ν

$p^\nu\varkappa 2$ . . *ſecundum* . ${}^\nu\mathfrak{A}\, a^{\nu-1}\mathfrak{a}^1 A$

$p^\nu\varkappa 3$ . . *tertium* . ${}^\nu\mathfrak{A}\, a^{\nu-1}\mathfrak{a}^2 A + {}^\nu\mathfrak{B}\, a^{\nu-2}\mathfrak{b}^2 B$

$p^\nu\varkappa 4$ . . *quartum* . ${}^\nu\mathfrak{A}\, a^{\nu-1}\mathfrak{a}^3 A + {}^\nu\mathfrak{B}\, a^{\nu-2}\mathfrak{b}^3 B + {}^\nu\mathfrak{C}\, a^{\nu-3}\mathfrak{c}^3 C$

. . . .

$p^\nu\varkappa(m+1)$ . . *(m+1)tum* . ${}^\nu\mathfrak{A}\, a^{\nu-1}\mathfrak{a}^m A + {}^\nu\mathfrak{B}\, a^{\nu-2}\mathfrak{b}^m B + {}^\nu\mathfrak{C}\, a^{\nu-3}\mathfrak{c}^m C \ldots + {}^\nu\mathfrak{M}\, a^{\nu-m}\mathfrak{m}^m \mathfrak{M}$

$p[a, b, c, d \ldots]$ $\begin{pmatrix} b, & c, & d, & e, & f, & \ldots \\ 1, & 2, & 3, & 4, & 5, & \ldots \end{pmatrix}$

Litera p ſcilicet feriem indicat, ν dignitatis ipſius exponentem, $\varkappa$ cum numero adpoſito coefficientem numero ipſi reſpondentem. Tale ſignum e. g. $p^\nu\varkappa(m+1)$ enunciatur hoc modo: *Seriei* p *ad dignitatem exponentis* ν *evectae coefficiens* (*m* + 1) *tus*. Hinc ſequitur:

$$p^\nu = p^\nu\varkappa 1\, z^{\nu\mu} + p^\nu\varkappa 2\, z^{\nu\mu+\beta} + p^\nu\varkappa 3\, z^{\nu\mu+2\beta} + p^\nu\varkappa 4\, z^{\nu\mu+3\beta} + \ldots\ldots + p^\nu\varkappa(m+1)\, z^{\nu\mu+m\beta} + \ldots\ldots$$
$$p[\,a,\ b,\ c,\ d,\ e \ldots\ldots\,]$$

Haec coefficientium notae dignitatis p^ν *Signa localia* vocantur, formulaeque ex illis compoſitae, *formulae locales* *). Signis et formulis localibus adpoſitum ſignum $p[a, b, c, d, e \ldots]$ dicitur *Scala* eaque ſimpliciter ſeriei datae p coefficientes ex ordine indicat. Scala igitur diligenter ab indice (§. II. Def. 1.) diſtinguenda eſt; hoc enim docetur, quos ad coefficientes diſcerptionum claſſes referantur **).

Scho-

*) Signa localia *coefficientium* et *terminorum integrorum* (ubi ponitur γ pro $\varkappa$, veluti $p^\nu\gamma(m+1)$) primus uſurpavit Hindenburgius (*Infinit. Dign.* p. 71, 3; p. 93 — 98; 136 — 141). Melior, quae hic adhibetur, notatio *Nov. Syſt.* p. XXXIII, 2. De ſignorum *localium* cum *combinatoriis* comparatione *Ib.* p. LI — LIII. *Id. Paral. ad Ser. Reverſ.* p. VIII.

**) Rothius, *Vir Clariſſimus,* in libello cui titulus: *Formulae de ſerierum reverſione demonſtratio univerſalis. Lipſiae* 1793. Scalas introduxit. Signa idem localia in hac diſſertatione ingenioſe adhibuit, earumque ope formulam de ſerierum reverſione combinatoriam, ab Eschenbachio (1789) propoſitam, primus et rigoroſe demonſtravit.

Digitized by Google

Scholion. Formula localis, exempli loco hic propoſita, *dignitatum Infinitinomii formula* appellatur *localis.* Haec quoque in tabula adjecta exprimitur.

* * *

Sed haec de Infinitinomio protuliſſe, ſufficiat. Pergimus jam ad ea explicanda, quae rem noſtram propius attingunt.

---

§. XIV.

Definitio.

Variationes ſummae propoſitae n nominantur diſcerptiones numeri *n* (§. II. Def. 6.) ſine diſcrimine omnes, ſive ſint rite ordinatae, ſive non.

§. XV.

Explicatio.

A

1	1	1	1	1
	2	1	1	1
	1	2	1	1
		3	1	1
	1	1	2	1
		2	2	1
		1	3	1
			4	1
	1	1	1	2
		2	1	2
		1	2	2
			3	2
		1	1	3
			2	3
			1	4
				5

In ſchemate appoſito A, Variationum ſummarum 1, 2, 3, 4, 5 (§. XIV.), Variationes ſummarum minorum a Variationibus ſummarum majorum ita involvuntur, ut illae ex his poſſint exſecari, id quod angulis interjectis docetur, quam ob rem ipſa haec variationum conſtructio *Involutio* vocatur *Combinatoria* *).

§. XVI.

Problema.

Ex involutione Variationum summae n (§. XV.) *conſtruere Involutionem Variationum ſummae* $(n+1)$.

Solu-

*) De Involutionum et Evolutionum (quae inter reliquas operationes combinatorias facile principatum obtinent) natura, diverſitate et in disquiſitionibus analyticis efficacia, ſumma et utilitate copioſe egit Hindenburgius, *Vir Excellentiſſimus*, (*Archiv der reinen und angewandten Mathematik H.* I. p. 13. ſeqq. *H.* II. III. et IV. *Programma* ſupra (§. IV. *) laudatum.)

SOLVTIO.

I. Variationi ſummae n cuilibet ad dextram unitas adponatur.

II. Earundem Variationum ſummae n elementa ad dextram extrema unitate augeantur, et, quae ita prodeunt, complexiones complexionibus per I. ortis verticaliter ſubſcribantur.

v. c. poſito $n = 1$,

datum eſt
```
1
```
hinc fit per. I
```
1 1
```
per I. II.
```
1 1
  2
```

poſito $n = 2$,

datum eſt
```
1 1
  2
```
hinc fit per I.
```
1 1 1
  2 1
```
per I. II.
```
1 1 1
  2 1
  1 2
    3
```

poſito $n = 3$,

datum eſt
```
1 1 1
  2 1
  1 2
    3
```
hinc fit per I.
```
1 1 1 1
  2 1 1
  1 2 1
    3 1
```
per I. II.
```
1 1 1 1
  2 1 1
  1 2 1
    3 1
  1 1 2
    2 2
    1 3
      4
```

DEMONSTRATIO.

Ponatur ſolutionem propoſitam problemati ſatisfacere, ſi quaeratur Involutio Variationum ſummarum 1, 2, 3, ... p; Variationes autem ſummae $(p + 1)$ regulis traditis non reperiri omnes, ſed unam vel plures deeſſe. Jam quaelibet deficientium Variationum ſummae $(p + 1)$ deſineret

vel in ipſam unitatem v. c. 321..... 121

vel in numerum unitate majorem v. c. 213...... 53.

Si *illud*, deficeret 321... 12, ſi *hoc*, deficeret 213.... 52 in Variationibus ſummae praecedentis p; id quod ſuppoſitioni repugnat. Eodem modo oſtenditur, eandem complexionem non poſſe ſaepius occurere. Conſtat autem, ſolutionem ſatisfacere problemati, poſito $p = 1$, et $p = 2$ itaque ſatisfacit etiam pro Variationibus ſummae cujuslibet $(p + 1) = 3; 4; 5; \ldots$ quaerendis.

§. XVII.

Digitized by Google

§. XVII.

Explicatio.

B.

1	1	1	1	1	1	1
	2	1	1	1	1	1
		3	1	1	1	1
		2	2	1	1	1
			4	1	1	1
			3	2	1	1
				5	1	1
			2	2	2	1
				4	2	1
				3	3	1
					6	1
				3	2	2
					5	2
					4	3
						7

In ſchemate B Combinationes ſummarum 1, 2, 3, 4, 5, 6, 7 ita exhibentur, ut elementorum ordo ſit inverſus ipſaeque Combinationes conſtituant *Involutionem Combinatoriam* (Cf. §. XV.)

§. XVIII.

Problema.

Ex Involutione Combinationum ſummae n §. XVII. conſtruere Involutionem Combinationum ſummae $(n+1)$.

Solutio.

I. Cuivis Combinationi ſummae n adponatur a dextra unitas,

II. Vnitate augeantur extrema ad dextram elementa earum complexionum, quae vel duobus terminantur elementis diverſis, vel unione (§. II. Def. 3.) conſtant, et quae ita prodeunt, complexiones complexionibus per I. ortis verticaliter ſubſcribantur,

v. c. poſito $n = 1$,

datum eſt

1	

hinc fit per. I

1	1

per I. II.

1	1
	2

poſito $n = 2$,

datum eſt

1	1
	2

hinc fit per I.

1	1	1
	2	1

per I. II.

1	1	1
	2	1
		3

poſito $n = 3$,

datum eſt

1	1	1
	2	1
		3

hinc fit per I.

1	1	1	1
	2	1	1
		3	1

per I. II.

1	1	1	1
	2	1	1
		3	1
		2	2
			4

Digitized by Google

Solutionis hujus demonſtratio eadem eſt, quae in §. XVI.

Scholion. Involutionum genus ab eo, quod §. XV — XVIII. propoſuimus, diverſum, hic vero non adhibitum, extat in *Infinit. Dign.* p. 79. 80, et p. 133.

§. XIX.

Explicatio.

Si *Variationes ſummarum* 1, 2, 3, 4, ... n exhibeantur involutorie (§. XVI.), et numerorum loco ponantur literae $a, b, c, d, \ldots n$, haec Involutio Variationum notetur ſigno:

$$\overset{n}{\mathfrak{J}}\binom{a,b,c,\ldots n}{1,2,3,\ldots n}$$

atque, ſi cuilibet complexioni numerus primo a laeva ſuorum elementorum reſpondens praefigatur, id Variationum genus ſummae n cum ſuis *Comitibus numericis* *) exprimatur ſigno:

$$\overset{n}{\mathfrak{i}\mathfrak{J}}\binom{a,b,c,\ldots n}{1,2,3,\ldots n}$$

Exempla. Pro ſummae n valoribus 1, 2, 3, 4, 5 .... prodeunt:

$$\overset{1}{\mathfrak{i}\mathfrak{J}}\binom{a}{1} = \underline{1a}|$$

Variationes ſummae 1 cum ſuis comitibus,

$$\overset{2}{\mathfrak{i}\mathfrak{J}}\binom{a,b}{1,2} = \begin{array}{r|l|} \underline{1a} & a \\ & \underline{2b} \end{array}$$

Variationes ſummae 2 cum ſuis comitibus,

$$\overset{3}{\mathfrak{i}\mathfrak{J}}\binom{a,b,c}{1,2,3} = \begin{array}{r|l|l|} \underline{1a} & a & a \\ & \underline{2b} & a \\ & 1a & b \\ & & \underline{3c} \end{array}$$

Variationes ſummae 3 cum ſuis comitibus,

$\overset{4}{\mathfrak{i}\mathfrak{J}} =$

*) Scilicet Hindenburgius, *V. Cel.* complexionum literalium *Comites numericos* ab ipſarum numeris permutationum ſive *coefficientibus polynomialibus* (Cf. §. VII. *) diſtinguit, et a quacunque polynomialium modificatione, aut alia lege oriundos numeros *Comites complexionum* nominat.

Digitized by Google

$$ {}^{4}\mathfrak{J}\binom{a,b,c,d}{1,\;2,\;3,\;4} = \begin{array}{r} 1a\,a\,a\,a \\ 2b\,a\,a \\ 1a\,b\,a \\ 3c\,a \\ 1a\,a\,b \\ 2b\,b \\ 1a\,c \\ 4d \end{array} $$

Variationes ſummae 4 cum ſuis comitibus,

$$ {}^{5}\mathfrak{J}\binom{a,b,c,d,e}{1,\;2,\;3,\;4,\;5} = \begin{array}{r} 1a\,a\,a\,a\,a \\ 2b\,a\,a\,a \\ 1a\,b\,a\,a \\ 3c\,a\,a \\ 1a\,a\,b\,a \\ 2b\,b\,a \\ 1a\,c\,a \\ 4d\,a \\ 1a\,a\,a\,b \\ 2b\,a\,b \\ 1a\,b\,b \\ 3c\,b \\ 1a\,a\,c \\ 2b\,c \\ 1a\,d \\ 5e \end{array} $$

Variationes ſummae 5 cum ſuis comitibus,

etc. etc. etc.

§. XX.

Explicatio.

Involutio Combinationum ſummae n ex elementis $a, b, c, d, \ldots . n$ (§. XVIII et §. VI.) notetur (*Arch. der Math. Heft. IV.* p. 417, 418) ſigno:

$$ {}^{n}\mathrm{J}\binom{a,b,c,..n}{1,\;2,\;3,\;..\;n} = {}^{n}\mathrm{A} + {}^{n}\mathrm{B} + {}^{n}\mathrm{C} + {}^{n}\mathrm{D} + \ldots\ldots + {}^{n}\mathcal{N} \binom{a,\,b,\,c,\,d,\ldots\ldots n}{1,\;2,\;3,\;4,\ldots\ldots n} $$

et,

Digitized by Google

et, ſi ſingulis complexionibus praefigatur numerus permutationum, ſigno:

$$\underset{\binom{a,b,c,\ldots n}{1,2,3\ldots n}}{\mathfrak{j}^n J} = \underset{\binom{a,b,c,d,\ldots,n}{1,2,3,4,\ldots n}}{\mathfrak{a}^n A + \mathfrak{b}^n B + \mathfrak{c}^n C + \mathfrak{d}^n D + \ldots\ldots \mathfrak{n}^n \mathcal{N}}$$

Si denique numeri Permutationum per ſummam n multiplicentur omnes, quilibet autem numero elementorum complexionis, cui adpoſitus eſt, dividatur, haec Involutio Combinationum ſummae n ex elementis $a, b, c, d, \ldots n$ cum *numericis Comitibus* ſuis (Cf. §. XIX. *) exprimatur ſigno:

$$\underset{\binom{a,b,c,\ldots n}{1,2,3\ldots n}}{\mathfrak{i}^n J} = \underset{\binom{a,\ b,\ c,\ d,\ e,\ \ldots\ldots n}{1,\ 2,\ 3,\ 4,\ \ldots\ldots\ldots n}}{\frac{n}{1}\mathfrak{a}^n A + \frac{n}{2}\mathfrak{b}^n B + \frac{n}{3}\mathfrak{c}^n C + \frac{n}{4}\mathfrak{d}^n D + \ldots + \frac{n}{m}\mathfrak{m}^n \mathcal{M} + \ldots + \frac{n}{n}\mathfrak{n}^n \mathcal{N}}$$

Exempla. Pro ſummae n valoribus 1, 2, 3, 4, 5, ... prodeunt:

$$\underset{\binom{a}{1}}{\mathfrak{i}^1 J} = \begin{array}{l} 1a \end{array} = \underset{\binom{a}{1}}{\frac{1}{1}\mathfrak{a}^1 A}$$

$$\underset{\binom{a,b}{1,2}}{\mathfrak{i}^2 J} = \begin{array}{l|l} 1a & a \\ & 2b \end{array} = \underset{\binom{a,\ b}{1,\ 2}}{\frac{2}{1}\mathfrak{a}^2 A + \frac{2}{2}\mathfrak{b}^2 B}$$

$$\underset{\binom{a,b,c}{1,2,3}}{\mathfrak{i}^3 J} = \begin{array}{l|l|l} 1a & a & a \\ & 3b & a \\ & & 3c \end{array} = \underset{\binom{a,\ b,\ c}{1,\ 2,\ 3}}{\frac{3}{1}\mathfrak{a}^3 A + \frac{3}{2}\mathfrak{b}^3 B + \frac{3}{3}\mathfrak{c}^3 C}$$

$$\underset{\binom{a,b,c,d}{1,2,3,4}}{\mathfrak{i}^4 J} = \begin{array}{l|l|l|l} 1a & a & a & a \\ & 4b & a & a \\ & & 4c & a \\ & & 2b & b \\ & & & 4d \end{array} = \underset{\binom{a,\ b,\ c,\ d}{1,\ 2,\ 3,\ 4}}{\frac{4}{1}\mathfrak{a}^4 A + \frac{4}{2}\mathfrak{b}^4 B + \frac{4}{3}\mathfrak{c}^4 C + \frac{4}{4}\mathfrak{d}^4 D}$$

$$\underset{\binom{a,b,c,d,e}{1,2,3,4,5}}{\mathfrak{i}^5 J} = \begin{array}{l|l|l|l|l} 1a & a & a & a & a \\ & 5b & a & a & a \\ & & 5c & a & a \\ & & 5b & b & a \\ & & & 5d & a \\ & & & 5c & b \\ & & & & 5e \end{array} = \underset{\binom{a,\ b,\ c,\ d,\ e}{1,\ 2,\ 3,\ 4,\ 5}}{\frac{5}{1}\mathfrak{a}^5 A + \frac{5}{2}\mathfrak{b}^5 B + \frac{5}{3}\mathfrak{c}^5 C + \frac{5}{4}\mathfrak{d}^5 D + \frac{5}{5}\mathfrak{e}^5 E}$$

Scholion

Digitized by Google

Scholion. Signa hic et §. XIX. exhibita ſunt *involutoria*, in quibus recta litera J; *Combinationum*, obliqua vero $\mathcal{J}$; *Variationum* involutoriam conſtructionem (§ XV. et XVII.) ſecundum indicem $\binom{a,\,b,\,c,\,d,\,e \ldots n}{1,\,2,\,3,\,4,\,5 \ldots n}$, litera n in fronte a laeva ſummam et germanicae minores $\mathfrak{i}$ et $\mathfrak{i}$ comites complexionum reſpondentes ſignificant.

§. XXI.

Theorema.

$$Log.\,nat.[1-(ax+bx^2+cx^3+\ldots)] = -\left(\frac{\mathfrak{i}^1 J x^1}{1}+\frac{\mathfrak{i}^2 J x^2}{2}+\frac{\mathfrak{i}^3 J x^3}{3}+\frac{\mathfrak{i}^4 J x^4}{4}\cdots+\frac{\mathfrak{i}^n J x^n}{n}+\cdots\right)\binom{a,\; b,\; c,\; d,\; \ldots\ldots n, \ldots}{1,\; 2,\; 3,\; 4,\; \ldots\ldots n, \ldots}$$

Demonstratio.

$$\text{Sit } Log.\,nat.[1-(ax+bx^2+cx^3\ldots)] = Log.\,nat.(1-y) = -\left(\frac{y}{1}+\frac{y^2}{2}+\frac{y^3}{3}\cdots+\frac{y^n}{n}\cdots\right)$$

unde, dignitatibus ſeriei y ſecundum §. VIII. expreſſis atque ita diſpoſitis, ut coefficientes eiusdem potentiae quantitatis x conſtituant ſeriem verticalem, prodit:

$$Log.\,nat.[1-(ax+bx^2+cx^3+\ldots)] = -\left[\begin{array}{l|l|l|l} \frac{\mathfrak{a}^1 A}{1}x+\frac{\mathfrak{a}^2 A}{2} & x^2+\frac{\mathfrak{a}^3 A}{3} & x^3+\cdots\cdots+\frac{\mathfrak{a}^n A}{n} & x^n+\cdots \\ \quad+\frac{\mathfrak{b}^2 B}{2} & \quad+\frac{\mathfrak{b}^3 B}{3} & \quad+\frac{\mathfrak{b}^n B}{n} & \\ & \quad+\frac{\mathfrak{c}^3 C}{3} & \quad+\frac{\mathfrak{c}^n C}{n} & \\ & & \quad+\frac{\mathfrak{m}^n \mathcal{M}}{m} & \\ & & \quad+\frac{\mathfrak{n}^n \mathcal{N}}{n} & \end{array}\right]\binom{a,\; b,\; c,\; d,\; e \ldots n, \ldots}{1,\; 2,\; 3,\; 4,\; 5 \ldots n, \ldots}$$

Coefficiens termini generalis $\left(\frac{\mathfrak{a}^n A}{1}+\frac{\mathfrak{b}^n B}{2}+\frac{\mathfrak{c}^n C}{3}+\cdots+\frac{\mathfrak{m}^n \mathcal{M}}{m}+\cdots+\frac{\mathfrak{n}^n \mathcal{N}}{n}\right)x^n$ ſecundum §. XX. aequalis eſt $\frac{\mathfrak{i}^n J}{n}$; itaque, valoribus 1, 2, 3, 4, 5, … ſucceſſive poſitis loco n, aequatio oritur Theorematis.

§. XXII.

Digitized by Google

§. XXII.

Theorema.

Eſt

$$\underset{(a)}{i^1\mathcal{J}} = 1a$$

$$\underset{(a,b)}{i^2\mathcal{J}} = \underset{(a)}{i^1\mathcal{J}a} + 2b$$

$$\underset{(a,b,c)}{i^3\mathcal{J}} = \underset{(a,\ b)}{i^2\mathcal{J}a + i^1\mathcal{J}b} + 3c$$

$$\underset{(a,b,c,d)}{i^4\mathcal{J}} = \underset{(a,\ b,\ c)}{i^3\mathcal{J}a + i^2\mathcal{J}b + i^1\mathcal{J}c} + 4d$$

$$\vdots$$

$$\underset{(a,b,c\ldots n)}{i^n\mathcal{J}} = \underset{\left(a,\ b,\ c,\ d,\ e\ \ldots\ \overset{-m}{n},\ \ldots\ \overset{-3}{n},\ \overset{-2}{n},\ \overset{-1}{n},\right)}{i^{n-1}\mathcal{J}a + i^{n-2}\mathcal{J}b + i^{n-3}\mathcal{J}c + \ldots + i^{n-m}\mathcal{J}m + \ldots + i^3\mathcal{J}\overset{-3}{n} + i^2\mathcal{J}\overset{-2}{n} + i^1\mathcal{J}\overset{-1}{n}} + nn$$

In his formulis elementa ſingula *a, b, c, d,* . . . ., ſignis involutoriis §. XIX. juncta, ad ſingulas complexiones, quas ſigna exprimunt, referenda eſſe, per ſe clarum eſt. Literae autem cum numeris ſupra poſitis deſignant,

$\overset{-1}{n}$ elementum $(n-1)$ tum,

$\overset{-2}{n}$ „ $(n-2)$ tum,

$\overset{-3}{n}$ „ $(n-3)$ tum,

$\vdots$

$\overset{-m}{n}$ „ $(n-m)$ tum,

ubi primum elementum eſt *a*, et *ntum n*; nempe numeri $-1, -2, -3, \ldots -m$, ſcripti ſupra *n*, *diſtantiam* exponunt *retrorſum* ab *nto*, et proinde *Diſtantiae Exponentes* vocantur. Hindenb. *Nov. Syſt.* p. XXXVII. ſeq. Toepf. *Comb. Anal.* p. 164. ſeq.

Demonstratio.

Theorematis ratio ſtatim ex ipſa conſtructione Variationum §. XVI. et explicatione §. XIX. liquet.

§. XXIII.

Theorema.

Elementis datae combinationis permutatis permutationumque alia alii ſubſcripta, uti mos eſt, ſeries elementorum verticalis quaelibet, toties unumquodque continet, quoties prima.

De-

Demonstratio.

Elementum quodvis toties eandem tenet ſedem, quoties reliqua permutari poſſunt, id ex ipſa liquet permutationum definitione (§. II. Def. 8), toties igitur in hac redit ſerie verticali, quoties in illa.

Corollarium I.

Significet $\mathfrak{m}$ numerum permutationum combinationis numericae ſummae n claſſis $m$tae, erit $n.\mathfrak{m}$ ſumma omnium numerorum, quibus permutationes illae conſtant, et cum numerus quilibet toties in hac recurrit ſerie verticali, quoties in illa, ſingulae ſeriei verticalis ſumma eſt $\frac{n}{m}.\mathfrak{m}$. Sic:

v. c. $\begin{bmatrix} 22233 \\ 22323 \\ 22332 \\ 23223 \\ 23232 \\ 23322 \\ 32223 \\ 32232 \\ 32322 \\ 33222 \end{bmatrix}$ ſunt permutationes combinationis numericae 22233 (§. II. Def. 8.) in qua $n = 12$, $m = 5$, et $\mathfrak{m} = 10$, unius itaque ſeriei verticalis ſumma eſt: $\frac{12}{5}.10 = 2+2+2+2+2+2+3+3+3+3$

Corollarium II.

Elementis datae combinationis Summae n, claſſis $m$tae et indicis $\begin{pmatrix} a, b, c, d, e, \ldots n \\ 1, 2, 3, 4, 5, \ldots n \end{pmatrix}$ permutatis, et praepoſito cuique permutationi numero, qui primo ad laevam ipſius elemento reſpondet, ea re productum data combinatione expreſſum multiplicatur per $\frac{n}{m}.\mathfrak{m}$ ſecundum Corollarium I. Sic data combinatione $bbbcc$ eſt $n = 12$, $m = 5$, $\mathfrak{m} = 10$.

atque $\begin{Bmatrix} 2bbbcc \\ 2bbcbc \\ 2bbccb \\ 2bcbbc \\ 2bcbcb \\ 2bccbb \\ 3cbbbc \\ 3cbbcb \\ 3cbcbb \\ 3ccbbb \end{Bmatrix} = \frac{12}{5}.10.\,bbbcc = 24\,bbbcc.$

§. XXIV.

Theorema.

$$n\,[\tfrac{1}{1}\mathfrak{a}^n\mathfrak{A} + \tfrac{1}{2}\mathfrak{b}^n\mathfrak{B} + \tfrac{1}{3}\mathfrak{c}^n\mathfrak{C} + \tfrac{1}{4}\mathfrak{d}^n\mathfrak{D} + \ldots + \tfrac{1}{m}\mathfrak{m}^n\mathcal{M} + \ldots \tfrac{1}{n}\mathfrak{n}^n\mathcal{N}] = \mathfrak{i}^n\mathfrak{J} = \mathfrak{i}^n\mathcal{J}$$
$$\begin{pmatrix} a, & b, & c, & d, & \ldots\ldots & n \\ 1, & 2, & 3, & 4, & \ldots\ldots & n \end{pmatrix} \qquad \begin{pmatrix} a, b, c, \ldots n \\ 1, 2, 3, \ldots n \end{pmatrix} \qquad \begin{pmatrix} a, b, c, \ldots n \\ 1, 2, 3 \ldots n \end{pmatrix}$$

De-

Digitized by Google

Demonstratio.

I. Ipſa liquet explicatione §. XX. eſſe

$$n\left(\frac{1}{1}a^nA+\frac{1}{2}b^nB+\frac{1}{3}c^nC+\frac{1}{4}d^nD+\ldots+\frac{1}{m}m^n\mathcal{M}+\ldots\frac{1}{n}n^n\mathcal{N}\right)_{\left(\begin{smallmatrix}a, & b, & c, & d, & \ldots & n\\ 1, & 2, & 3, & 4, & \ldots & n\end{smallmatrix}\right)} = 1^n\mathfrak{J}_{\left(\begin{smallmatrix}a, & b, & c, & \ldots & n\\ 1, & 2, & 3, & \ldots & n\end{smallmatrix}\right)}$$

II. Si pro Combinationibus ſummae n, Claſſis $m$tae cum numeris permutationum ſuarum i.e. pro:

$$m^n\mathcal{M}_{\left(\begin{smallmatrix}a, & b, & c, & \ldots & n\\ 1, & 2, & 3, & \ldots & n\end{smallmatrix}\right)}$$

ipſae harum Combinationum ſubſtituantur permutationes (i.e. Variationum ſummae n Claſſis $m$ta Def. §. XIV.) et cuique permutationi praefigatur numerus primo elementorum ſuorum reſpondens; eo producta ſigno:

$${}^n\mathcal{M}_{\left(\begin{smallmatrix}a, & b, & c, & \ldots & n\\ 1, & 2, & 3, & \ldots & n\end{smallmatrix}\right)}$$

expreſſa ſecundum Corollarium II. §. XXIII. multiplicantur per $\frac{n}{m}.m$ (valor ſcilicet numeri permutationum m pendet a diverſitate factorum cujusvis illorum productorum v. §. VII.*)

Itaque
$$\frac{n}{1}a^nA+\frac{n}{2}b^nB+\frac{n}{3}c^nC+\frac{n}{4}d^nD+\ldots+\frac{n}{m}m^n\mathcal{M}+\ldots+\frac{n}{n}n^n\mathcal{N}$$
$$\left(\begin{smallmatrix}a, & b, & c, & d, & e, & \ldots & n\\ 1, & 2, & 3, & 4, & 5, & \ldots & n\end{smallmatrix}\right)$$

Variationibus exprimitur, ſi prima, ſecunda, tertia, verbo Claſſes omnes Variationum ſummae n conſtruuntur et cuique variationi proponitur numerus primo ſuorum elementorum reſpondens. His variationibus autem involutorie diſpoſitis oritur (§. XIX.):

$$1^n\mathfrak{J}_{\left(\begin{smallmatrix}a, & b, & c, & \ldots & n\\ 1, & 2, & 3, & \ldots & n\end{smallmatrix}\right)} = \frac{n}{1}a^nA+\frac{n}{2}b^nB+\frac{n}{3}c^nC+\frac{n}{4}d^nD+\ldots+\frac{n}{m}m^n\mathcal{M}+\ldots+\frac{n}{n}n^n\mathcal{N}_{\left(\begin{smallmatrix}a, & b, & c, & d, & \ldots & n\\ 1, & 2, & 3, & 4, & \ldots & n\end{smallmatrix}\right)}$$

Supereſt ut exempla afferantur:

Poſito $n=1$, eſt $\underline{1a|} = \underline{1a|} = 1\left(\frac{1}{1}a^1A\right)_{\left(\begin{smallmatrix}a\\1\end{smallmatrix}\right)}$

$n=2$, „ $\begin{array}{l|l|}1a & a\\ \underline{2b} & \end{array} = \begin{array}{l|l|}1a & a\\ \underline{2b} & \end{array} = 2\left(\frac{1}{1}a^2A+\frac{1}{2}b^2B\right)_{\left(\begin{smallmatrix}a, & b,\\ 1, & 2,\end{smallmatrix}\right)}$

$n=3$, „ $\begin{array}{l|l|l|}1a & a & a\\ \underline{2b} & a & \\ 1a\ b & & \\ \underline{3c} & & \end{array} = \begin{array}{l|l|l|}1a & a & a\\ \underline{3b} & a & \\ \underline{3c} & & \end{array} = 3\left(\frac{1}{1}a^3A+\frac{1}{2}b^3B+\frac{1}{3}c^3C\right)_{\left(\begin{smallmatrix}a, & b, & c,\\ 1, & 2, & 3\end{smallmatrix}\right)}$

Po-

Digitized by Google

Poſito $n = 4$ eſt $\begin{array}{l} 1\underline{a}|a|a|a| \\ \underline{2b}|a|a| \\ 1a\ b|a| \\ \underline{3c}|a| \\ 1a\ a\ b| \\ 2b\ b| \\ 1a\ c| \\ \underline{4d}| \end{array} = \begin{array}{l} 1\underline{a}|a|a|a| \\ \underline{4b}|a|a| \\ \underline{4c}|a| \\ 2b\ b| \\ \underline{4d}| \end{array} = 4\left(\frac{1}{1}\mathfrak{a}^{4}\mathrm{A} + \frac{1}{2}\mathfrak{b}^{4}\mathrm{B} + \frac{1}{3}\mathfrak{c}^{4}\mathrm{C} + \frac{1}{4}\mathfrak{d}^{4}\mathrm{D}\right)_{(a_1,\ b_2,\ c_3,\ d_4)}$

§. XXV

Explicatio.

Ponatur:

$${}^{1}\Sigma = \mathrm{i}^{1}\mathrm{J}_{(a)}\,\alpha + \mathrm{i}^{1}\mathrm{J}_{(\mathfrak{a})}\,\beta + \ldots + \mathrm{i}^{1}\mathrm{J}_{(\mathfrak{A})}\,\varrho = 1\underline{a}|\alpha + 1\underline{\mathfrak{a}}|\beta + \ldots + 1\underline{\mathfrak{A}}|\varrho$$

$${}^{2}\Sigma = \mathrm{i}^{2}\mathrm{J}_{(a,b)}\,\alpha + \mathrm{i}^{2}\mathrm{J}_{(\mathfrak{a},\mathfrak{b})}\,\beta + \ldots + \mathrm{i}^{2}\mathrm{J}_{(\mathfrak{A},\mathfrak{B})}\,\varrho = \begin{array}{l} 1\underline{a}|a|\alpha \\ \underline{2b}| \end{array} + \begin{array}{l} 1\underline{\mathfrak{a}}|\mathfrak{a}|\beta \\ \underline{2\mathfrak{b}}| \end{array} + \ldots + \begin{array}{l} 1\underline{\mathfrak{A}}|\mathfrak{A}|\varrho \\ \underline{2\mathfrak{B}}| \end{array}$$

$${}^{3}\Sigma = \mathrm{i}^{3}\mathrm{J}_{(a,b,c)}\,\alpha + \mathrm{i}^{3}\mathrm{J}_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})}\,\beta + \ldots + \mathrm{i}^{3}\mathrm{J}_{(\mathfrak{A},\mathfrak{B},\mathfrak{C})}\,\varrho = \begin{array}{l} 1\underline{a}|a|a|\alpha \\ \underline{3b}|a| \\ \underline{3c}| \end{array} + \begin{array}{l} 1\underline{\mathfrak{a}}|\mathfrak{a}|\mathfrak{a}|\beta \\ \underline{3\mathfrak{b}}|\mathfrak{a}| \\ \underline{3\mathfrak{c}}| \end{array} + \ldots + \begin{array}{l} 1\underline{\mathfrak{A}}|\mathfrak{A}|\mathfrak{A}|\varrho \\ \underline{3\mathfrak{B}}|\mathfrak{A}| \\ \underline{3\mathfrak{C}}| \end{array}$$

$${}^{n}\Sigma = \mathrm{i}^{n}\mathrm{J}_{(a,b,\ldots n)}\,\alpha + \mathrm{i}^{n}\mathrm{J}_{(\mathfrak{a},\mathfrak{b},\ldots \mathfrak{n})}\,\beta + \ldots + \mathrm{i}^{n}\mathrm{J}_{(\mathfrak{A},\mathfrak{B},\ldots \mathfrak{N})}\,\varrho$$

(Conf. Explicationem §. XX.)

§. XXVI.

Theorema.

$$Log.\,nat.\,[1-(ax+bx^{2}\ldots)]^{\alpha}[1-(\mathfrak{a}x+\mathfrak{b}x^{2}\ldots)]^{\beta}\ldots[1-(\mathfrak{A}x+\mathfrak{B}x^{2}\ldots)]^{\varrho} = \left(\frac{{}^{1}\Sigma}{1}x + \frac{{}^{2}\Sigma}{2}x^{2} + \ldots + \frac{{}^{n}\Sigma}{n}x^{n}\ldots\right)$$

Demon-

Digitized by Google

Demonstratio.

Eſt enim (§. XXI.), ſi cuique termino index adponatur:

$$\alpha\, Log.\, nat.\left[1-(ax+bx^2+cx^3+..)\right]=-\left(\underset{(a)}{\mathrm{i}^{1}\mathrm{J}\alpha}\frac{x}{1}+\underset{(a,b)}{\mathrm{i}^{2}\mathrm{J}\alpha}\frac{x^2}{2}+\underset{(a,b,c)}{\mathrm{i}^{3}\mathrm{J}\alpha}\frac{x^3}{3}+\ldots+\underset{(a,b,..n)}{\mathrm{i}^{n}\mathrm{J}\alpha}\frac{x^n}{n}+\ldots\right)$$

$$\beta\, Log.\, nat.\left[1-(\mathfrak{a}x+\mathfrak{b}x^2+\mathfrak{c}x^3+..)\right]=-\left(\underset{(\mathfrak{a})}{\mathrm{i}^{1}\mathrm{J}\beta}\frac{x}{1}+\underset{(\mathfrak{a},\mathfrak{b})}{\mathrm{i}^{2}\mathrm{J}\beta}\frac{x^2}{2}+\underset{(\mathfrak{a},\mathfrak{b},\mathfrak{c})}{\mathrm{i}^{3}\mathrm{J}\beta}\frac{x^3}{3}+\ldots+\underset{(\mathfrak{a},\mathfrak{b},..\mathfrak{n})}{\mathrm{i}^{n}\mathrm{J}\beta}\frac{x^n}{n}+\ldots\right)$$

. . . .

$$\varrho\, Log.\, nat.\left[1-(\mathfrak{A}x+\mathfrak{B}x^2+\mathfrak{C}x^3+..)\right]=-\left(\underset{(\mathfrak{A})}{\mathrm{i}^{1}\mathrm{J}\varrho}\frac{x}{1}+\underset{(\mathfrak{A},\mathfrak{B})}{\mathrm{i}^{2}\mathrm{J}\varrho}\frac{x^2}{2}+\underset{(\mathfrak{A},\mathfrak{B},\mathfrak{C})}{\mathrm{i}^{3}\mathrm{J}\varrho}\frac{x^3}{3}+\ldots+\underset{(\mathfrak{A},\mathfrak{B},..\mathfrak{N})}{\mathrm{i}^{n}\mathrm{J}\varrho}\frac{x^n}{n}+\ldots\right)$$

unde, ſi coefficientes earundem potentiarum quantitatis x per omnes ſeries verticales colligantur et ſignis §. XXV. exprimantur, formula prodit propoſita.

§. XXVII.

Haec ubi praemiſſa ſunt, ad ipſius libelli ſummam convertimur. In aequatione ſcilicet ſupra (§. I.) propoſita:

$$\left[1-(ax+bx^2\ldots)\right]^{\alpha}\left[1-(\mathfrak{a}x+\mathfrak{b}x^2..)\right]^{\beta}\ldots\left[1-(\mathfrak{A}x+\mathfrak{B}x^2..)\right]^{\varrho}=1-(\mathrm{A}x+\mathrm{B}x^2+\mathrm{C}x^3..+\mathrm{N}x^n\ldots)$$

ubi numerus factorum indeterminatus eſt atque Coefficientium et Exponentium valores quicunque eſſe poſſunt,

I. Summa Involutionum $\underset{(a,b,c..n)}{\mathrm{i}^{n}\mathrm{J}\alpha}+\underset{(\mathfrak{a},\mathfrak{b},\mathfrak{c},..\mathfrak{n})}{\mathrm{i}^{n}\mathrm{J}\beta}\ldots+\underset{(\mathfrak{A},\mathfrak{B},\mathfrak{C}..\mathfrak{N})}{\mathrm{i}^{n}\mathrm{J}\varrho}$; per Coefficientes A, B, C, .. N;

II. Coefficientes A, B, C, D, E, F, ... N, per Exponentes α, β, γ, δ, ε, ζ, ... ϱ; et Coefficientes (a, b, c, d, . .); ($\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, . . .); . . . ($\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, . . .);

III. Exponentes $\alpha, \beta, \gamma, \delta, ..\varrho$, per Coefficientes $(a, b, c, d, \ldots)$; $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, ..)$; .. $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \ldots)$; et A, B, C, D, . . . exprimentur.

Harum relationum prima et ſecunda, quibus multa communia ſunt, proximis duobus paragraphis afferuntur, de tertia vero ab illis longe diverſa, problemate de Incognitarum Eliminatione interjecto, actum eſt §. XXXI.

Digitized by Google

§. XXVIII.

Theorema.

Data aequatione §. XXVII. expoſita:

$$[1-(ax+bx^2..)]^{\alpha}[1-(\mathfrak{a}x+\mathfrak{b}x^2..)]^{\beta}..[1-(\mathfrak{A}x+\mathfrak{B}x^2..)]^{\varrho}=1-(\text{A}x+\text{B}x^2+\text{C}x^3\ldots+\text{N}x^n\ldots)$$

ſequitur:

$$I.\ {}^{n}\Sigma = \underset{(\text{A},\text{B},\text{C},..\text{N})}{i^{n}J} = \underset{(a,b,..n)}{i^{n}J\alpha} + \underset{(\mathfrak{a},\mathfrak{b},..\mathfrak{n})}{i^{n}J\beta} + \ldots\ldots + \underset{(\mathfrak{A},\mathfrak{B},..\mathfrak{N})}{i^{n}J\varrho}.$$

$$II.\ {}^{n}\Sigma = {}^{n-1}\Sigma\text{A} + {}^{n-2}\Sigma\text{B} + {}^{n-3}\Sigma\text{C} + \ldots + {}^{n-m}\Sigma\text{M} + \ldots + {}^{1}\Sigma\overset{-1}{\text{N}} + n\,\text{N}.$$

$$= \begin{cases} +\alpha\,\underset{(a,\ b,\ c,\ \ldots\ldots\ \overset{-1}{n})}{[i^{n-1}\mathfrak{J}a + i^{n-2}\mathfrak{J}b + i^{n-3}\mathfrak{J}c + \ldots + i^{n-m}\mathfrak{J}m + \ldots + i^{1}\mathfrak{J}\overset{-1}{n} + n\,n]} \\ +\beta\,\underset{(\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \ldots\ldots\ \overset{-1}{\mathfrak{n}})}{[i^{n-1}\mathfrak{J}\mathfrak{a} + i^{n-2}\mathfrak{J}\mathfrak{b} + i^{n-3}\mathfrak{J}\mathfrak{c} + \ldots + i^{n-m}\mathfrak{J}\mathfrak{m} + \ldots + i^{1}\mathfrak{J}\overset{-1}{\mathfrak{n}} + n\,\mathfrak{n}]} \\ \cdot \\ \cdot \\ +\varrho\,\underset{(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \ldots\ldots\ \overset{-1}{\mathfrak{N}})}{[i^{n-1}\mathfrak{J}\mathfrak{A} + i^{n-2}\mathfrak{J}\mathfrak{B} + i^{n-3}\mathfrak{J}\mathfrak{C} + \ldots + i^{n-m}\mathfrak{J}\mathfrak{M} + \ldots + i^{1}\mathfrak{J}\overset{-1}{\mathfrak{N}} + n\,\mathfrak{N}]} \end{cases}$$

Demonstratio.

Secundum §. XXI. eſt:

$$Log.\,nat.\,[1-(\text{A}x+\text{B}x^2+\text{C}x^3+\text{D}x^4..+\text{N}x^n..)] = -\underset{\left(\begin{smallmatrix}\text{A}, & \text{B}, & \text{C}, & \text{D}, & \ldots & \text{N}, & .. \\ 1, & 2, & 3, & 4, & \ldots & n, & ..\end{smallmatrix}\right)}{\left(\frac{i^{1}Jx^{1}}{1}+\frac{i^{2}Jx^{2}}{2}+\frac{i^{3}Jx^{3}}{3}\ldots+\frac{i^{n}Jx^{n}}{n}..\right)}$$

et ſecundum §. XXVI.

$$Log.\,nat.\,[1-(ax+bx^2..)]^{\alpha}[1-(\mathfrak{a}x+\mathfrak{b}x^2..)]^{\beta}..[1-(\mathfrak{A}x+\mathfrak{B}x^2..)]^{\varrho} = -\left[\frac{{}^{1}\Sigma x}{1}+\frac{{}^{2}\Sigma x^2}{2}\ldots+\frac{{}^{n}\Sigma x^n}{n}..\right]$$

Itaque, utriusque ſeriei coefficientibus comparatis, ſequitur:

$$I.\ {}^{n}\Sigma = \underset{(\text{A},\text{B},\text{C},..\text{N})}{i^{n}J} = \underset{(a,b,..n)}{i^{n}J\alpha} + \underset{(\mathfrak{a},\mathfrak{b},..\mathfrak{n})}{i^{n}J\beta} + \ldots + \underset{(\mathfrak{A},\mathfrak{B},..\mathfrak{N})}{i^{n}J\varrho},$$

Jam ſi adhibentur Combinationum loco Variationes, uti traditum eſt §. XXIV. fit:

$${}^{n}\Sigma = \underset{(\text{A},\text{B},\text{C},.\text{N})}{i^{n}\mathfrak{J}} = \underset{(a,b,..n)}{i^{n}\mathfrak{J}\alpha} + \underset{(\mathfrak{a},\mathfrak{b},..\mathfrak{n})}{i^{n}\mathfrak{J}\beta} + \ldots + \underset{(\mathfrak{A},\mathfrak{B},..\mathfrak{N})}{i^{n}\mathfrak{J}\varrho}$$

five

Digitized by Google

ſive, ſignis involutoriis ſecundum §. XXII. evolutis:

$$^{n}\Sigma = \underset{(A,\ B,\ C,\ \ldots\ldots\ \overset{n-1}{N})}{i^{n-1}\mathfrak{J}A + i^{n-1}\mathfrak{J}B + i^{n-1}\mathfrak{J}C + \ldots + i^{n-m}\mathfrak{J}M + \ldots + i^{1}\mathfrak{J}\overset{n-1}{N}} + nN$$

cumque ex formula *I* et §. XXIV. poſito $(n-m)$ loco (n), ſequatur:

$$^{n-m}\Sigma = \underset{(A,B,C..)}{i^{(n-m)}J} = \underset{(A,B,C..)}{i^{n-m}\mathfrak{J}}.$$

Formula prodit univerſalis, terminis recurrentibus compoſita:

$$II.\ ^{n}\Sigma = {}^{n-1}\Sigma A + {}^{n-1}\Sigma B + {}^{n-1}\Sigma C + \ldots + {}^{n-m}\Sigma M + \ldots + {}^{1}\Sigma \overset{n-1}{N} + nN$$

$$= \left\{\begin{array}{l} + \alpha\,[\underset{(a,\ b,\ c,\ \ldots\ldots\ \overset{n-1}{n})}{i^{n-1}\mathfrak{J}a + i^{n-1}\mathfrak{J}b + i^{n-1}\mathfrak{J}c + \ldots + i^{n-m}\mathfrak{J}m + \ldots + i^{1}\mathfrak{J}\overset{n-1}{n}} + nn] \\ + \beta\,[\underset{(\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \ldots\ldots\ \overset{n-1}{\mathfrak{n}})}{i^{n-1}\mathfrak{J}\mathfrak{a} + i^{n-1}\mathfrak{J}\mathfrak{b} + i^{n-1}\mathfrak{J}\mathfrak{c} + \ldots + i^{n-m}\mathfrak{J}\mathfrak{m} + \ldots + i^{1}\mathfrak{J}\overset{n-1}{\mathfrak{n}}} + n\mathfrak{n}] \\ \quad\vdots \\ + \varrho\,[\underset{(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \ldots\ldots\ \overset{n-1}{\mathfrak{N}})}{i^{n-1}\mathfrak{J}\mathfrak{A} + i^{n-1}\mathfrak{J}\mathfrak{B} + i^{n-1}\mathfrak{J}\mathfrak{C} + \ldots + i^{n-m}\mathfrak{J}\mathfrak{M} + \ldots + i^{1}\mathfrak{J}\overset{n-1}{\mathfrak{N}}} + n\mathfrak{N}] \end{array}\right.$$

Ex formula *I.* ſequitur, poſito $n = 1;\ 2;\ 3;\ 4;$

$$^{1}\Sigma = 1A| = 1a|\,\alpha + 1\mathfrak{a}|\,\beta + \ldots\ldots\ldots + 1\mathfrak{A}|\,\varrho$$

$$^{2}\Sigma = \begin{array}{cc} 1A| & A| \\ & 2B| \end{array} = \begin{array}{cc} 1a| & a| \\ & 2b| \end{array}\alpha + \begin{array}{cc} 1\mathfrak{a}| & \mathfrak{a}| \\ & 2\mathfrak{b}| \end{array}\beta + \ldots\ldots\ldots + \begin{array}{cc} 1\mathfrak{A}| & \mathfrak{A}| \\ & 2\mathfrak{B}| \end{array}\varrho$$

$$^{3}\Sigma = \begin{array}{ccc} 1A| & A| & A| \\ & 3B| & A| \\ & & 3C| \end{array} = \begin{array}{ccc} 1a| & a| & a| \\ & 3b| & a| \\ & & 3c| \end{array}\alpha + \begin{array}{ccc} 1\mathfrak{a}| & \mathfrak{a}| & \mathfrak{a}| \\ & 3\mathfrak{b}| & \mathfrak{a}| \\ & & 3\mathfrak{c}| \end{array}\beta + \ldots\ldots + \begin{array}{ccc} 1\mathfrak{A}| & \mathfrak{A}| & \mathfrak{A}| \\ & 3\mathfrak{B}| & \mathfrak{A}| \\ & & 3\mathfrak{C}| \end{array}\varrho$$

$$^{4}\Sigma = \begin{array}{cccc} 1A| & A| & A| & A| \\ & 4B| & A| & A| \\ & & 4C| & A| \\ & & 2B & B \\ & & & 4D| \end{array} = \begin{array}{cccc} 1a| & a| & a| & a| \\ & 4b| & a| & a| \\ & & 4c| & a| \\ & & 2b & b \\ & & & 4d| \end{array}\alpha + \begin{array}{cccc} 1\mathfrak{a}| & \mathfrak{a}| & \mathfrak{a}| & \mathfrak{a}| \\ & 4\mathfrak{b}| & \mathfrak{a}| & \mathfrak{a}| \\ & & 4\mathfrak{c}| & \mathfrak{a}| \\ & & 2\mathfrak{b} & \mathfrak{b} \\ & & & 4\mathfrak{d}| \end{array}\beta + \ldots + \begin{array}{cccc} 1\mathfrak{A}| & \mathfrak{A}| & \mathfrak{A}| & \mathfrak{A}| \\ & 4\mathfrak{B}| & \mathfrak{A}| & \mathfrak{A}| \\ & & 4\mathfrak{C}| & \mathfrak{A}| \\ & & 2\mathfrak{B} & \mathfrak{B} \\ & & & 4\mathfrak{D}| \end{array}\varrho$$

 Ex-

Digitized by Google

Ex formula *II.* vero ſequitur, poſito $n = 1;\ 2;\ 3;\ 4;$

$$^1\Sigma = 1\text{A} = (1a)\alpha + (1\mathfrak{a})\beta + \dots\dots + (1\mathfrak{A})\varrho$$

$$^2\Sigma = {}^1\Sigma\text{A} + 2\text{B} = (1'\underset{(a)}{\mathfrak{J}}a + 2b)\alpha + (1'\underset{(\mathfrak{a})}{\mathfrak{J}}\mathfrak{a} + 2\mathfrak{b})\beta + \dots\dots + (1'\underset{(\mathfrak{A})}{\mathfrak{J}}\mathfrak{A} + 2\mathfrak{B})\varrho$$

$$^3\Sigma = {}^2\Sigma\text{A} + {}^1\Sigma\text{B} + 3\text{C} = (1'\mathfrak{J}a + 1'\mathfrak{J}b + 3c)\alpha + (1'\mathfrak{J}\mathfrak{a} + 1'\mathfrak{J}\mathfrak{b} + 3\mathfrak{c})\beta + \dots + (1'\mathfrak{J}\mathfrak{A} + 1'\mathfrak{J}\mathfrak{B} + 3\mathfrak{C})\varrho$$
$$\qquad\qquad (a,\ b,)\qquad\qquad (\mathfrak{a},\ \mathfrak{b})\qquad\qquad (\mathfrak{A},\ \mathfrak{B})$$

$$^4\Sigma = {}^3\Sigma\text{A} + {}^2\Sigma\text{B} + {}^1\Sigma\text{C} + 4\text{D} = (1'\mathfrak{J}a + 1'\mathfrak{J}b + 1'\mathfrak{J}c + 4d)\alpha + \dots + (1'\mathfrak{J}\mathfrak{A} + 1'\mathfrak{J}\mathfrak{B} + 1'\mathfrak{J}\mathfrak{C} + 4\mathfrak{D})\varrho$$
$$\qquad\qquad (a,\ b,\ c)\qquad\qquad (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$$

§. XXIX.

Theorema.

Ex aequatione §. XXVII. *propoſita*

$$[1-(ax+bx^2..)]^\alpha.[1-(\mathfrak{a}x+\mathfrak{b}x^2..)]^\beta..[1-(\mathfrak{A}x+\mathfrak{B}x^2..)]^\varrho = 1-(\text{A}x+\text{B}x^2+\text{C}x^3..+\text{N}x^n..)$$

ſequitur:

$$\text{N} = \frac{1}{1}\mathfrak{a}^n\text{A} - \frac{1}{1.2}\mathfrak{b}^n\text{B} + \frac{1}{1.2.3}\mathfrak{c}^n\text{C} - \dots \pm \frac{1}{1.2.3\dots m}\mathfrak{m}^n\mathfrak{M} \mp \dots \pm \frac{1}{1.2\dots n}\mathfrak{n}^n\mathfrak{N}$$
$$\left(\frac{^1\Sigma}{1},\ \frac{^2\Sigma}{2},\ \frac{^3\Sigma}{3},\ \frac{^4\Sigma}{4},\dots\dots\frac{^n\Sigma}{n}\right)$$
$$1,\quad 2,\quad 3,\quad 4,\dots\dots n$$

Demonstratio.

Logarithmus naturalis factorum ad ſiniſtram in aequatione eſt (§. XXVI.):

$$-\left(\frac{^1\Sigma}{1}x^1 + \frac{^2\Sigma}{2}x^2 + \frac{^3\Sigma}{3}x^3 + \frac{^4\Sigma}{4}x^4 + \dots + \frac{^n\Sigma}{n}x^n + \dots\right)$$

cui aequale ſit $-\lambda$. Jam ſi h baſin logarithmorum naturalium denotat, conſtat eſſe:

$$h^{-\lambda} = 1 - \frac{1}{1}\lambda + \frac{1}{1.2}\lambda^2 - \frac{1}{1.2.3}\lambda^3 + \dots \mp \frac{1}{1.2.3\dots n}\lambda^n \pm \dots$$

ex qua aequatione, valore quantitatis $-\lambda$ reſtituto et dignitatibus eiusdem ſecundum §. VIII. expreſſis, ſequitur:

h

Digitized by Google

$$h^{-[\frac{{}^1\Sigma}{1}x+\frac{{}^2\Sigma}{2}x^2\ldots]} = 1-\mathfrak{a}\,{}^1\!A x - \begin{array}{l|l|l|l} \frac{1}{1}\mathfrak{a}\,{}^2\!A & x^2-\frac{1}{1}\mathfrak{a}\,{}^3\!A & x^3-\ldots-\frac{1}{1}\mathfrak{a}\,{}^n\!A & x^n\ldots \\ +\frac{1}{1.2}\mathfrak{b}\,{}^2\!B & +\frac{1}{1.2}\mathfrak{b}\,{}^3\!B & +\quad +\frac{1}{1.2}\mathfrak{b}\,{}^n\!B & \\ & -\frac{1}{1.2.3}\mathfrak{c}\,{}^3C & -\quad -\frac{1}{1.2.3}\mathfrak{c}\,{}^nC & \\ & & \mp\frac{1}{1.2\ldots n}\mathfrak{n}\,{}^n\mathcal{N} & \end{array} \quad [Q$$

$$\left(\underset{1,}{\frac{{}^1\Sigma}{1}},\underset{2,}{\frac{{}^2\Sigma}{2}},\underset{3,}{\frac{{}^3\Sigma}{3}},\ldots\underset{n}{\frac{{}^n\Sigma}{n}}\ldots\right)$$

Eſt vero $1-[Ax+Bx^2+Cx^3+\ldots+Nx^n\ldots]=h^{Log.nat.[1-(ax+bx^2\ldots)]^{\alpha}\ldots[1-(\mathfrak{A}x+\mathfrak{B}x^2\ldots)]^{\rho}}$

$$=h^{-[\frac{{}^1\Sigma x}{1}+\frac{{}^2\Sigma x^2}{2}+\frac{{}^3\Sigma x^3}{3}\ldots+\frac{{}^n\Sigma x^n}{n}\ldots]}$$

Itaque, ſerierum $1-[Ax+Bx^2+Cx^3+\ldots+Nx^n\ldots]$ et Q; comparatione inſtituta, reperitur:

$$N=\frac{1}{1}\mathfrak{a}\,{}^nA-\frac{1}{1.2}\mathfrak{b}\,{}^nB+\frac{1}{1.2.3}\mathfrak{c}\,{}^nC-\frac{1}{1.2.3.4.}\mathfrak{d}\,{}^nD+\ldots\pm\frac{1}{1.2\ldots m}\mathfrak{m}\,{}^n\mathcal{M}\mp\ldots\pm\frac{1}{1.2\ldots n}\mathfrak{n}\,{}^n\mathcal{N}$$

$$\left(\underset{1,}{\frac{{}^1\Sigma}{1}},\underset{2,}{\frac{{}^2\Sigma}{2}},\underset{3,}{\frac{{}^3\Sigma}{3}},\ldots\underset{n,}{\frac{{}^n\Sigma}{n}}\right)$$

Sic eſt, poſito $n=1;\ 2;\ 3;\ 4;$

$$A=\frac{1}{1}\,\frac{{}^1\Sigma}{1}$$

$$B=\frac{1}{1}\,\frac{{}^2\Sigma}{2}-\frac{1}{1.2}\,\frac{{}^1\Sigma}{1}\,\frac{{}^1\Sigma}{1}$$

$$C=\frac{1}{1}\,\frac{{}^3\Sigma}{3}-\frac{1}{1.2}\,2\frac{{}^1\Sigma}{1}\,\frac{{}^2\Sigma}{2}+\frac{1}{1.2.3.}\,\frac{{}^1\Sigma}{1}\,\frac{{}^1\Sigma}{1}\,\frac{{}^1\Sigma}{1}$$

$$D=\frac{1}{1}\,\frac{{}^4\Sigma}{4}-\frac{1}{1.2}\left[\begin{array}{l}2\frac{{}^1\Sigma}{1}\,\frac{{}^3\Sigma}{3}\\ \frac{{}^2\Sigma}{2}\,\frac{{}^2\Sigma}{2}\end{array}\right]+\frac{1}{1.2.3.}\,3\frac{{}^1\Sigma}{1}\,\frac{{}^1\Sigma}{1}\,\frac{{}^2\Sigma}{2}-\frac{1}{1.2.3.4.}\,\frac{{}^1\Sigma}{1}\,\frac{{}^1\Sigma}{1}\,\frac{{}^1\Sigma}{1}\,\frac{{}^1\Sigma}{1}.$$

§. XXX.

Digitized by Google

§. XXX.

P r o b l e m a.

Datis aequationibus numero r

$$
\begin{aligned}
{}^{1}s &= {}^{1}a\alpha + {}^{1}b\beta + {}^{1}c\gamma + \ldots\ldots + {}^{1}r\varrho \\
{}^{2}s &= {}^{2}a\alpha + {}^{2}b\beta + {}^{2}c\gamma + \ldots\ldots + {}^{2}r\varrho \\
{}^{3}s &= {}^{3}a\alpha + {}^{3}b\beta + {}^{3}c\gamma + \ldots\ldots + {}^{3}r\varrho \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
{}^{r}s &= {}^{r}a\alpha + {}^{r}b\beta + {}^{r}c\gamma + \ldots\ldots + {}^{r}r\varrho \;{}^{*)}
\end{aligned}
$$

valores quantitatum incognitarum $\alpha, \beta, \gamma, \ldots\ldots \varrho$ *eliminando quaerere.*

S o l u t i o.

Pars prior. Reperitur α regulis, quae ſequuntur:

1) Complexio *abcd* .... *r* toties in columna verticali ſcribatur, quoties elementa, 1, 2, 3, 4 .... *r* permutari poſſunt (Cf. §. VII*).

2) In hac Columna (1) literis ſingulis Complexionis primae adſcribantur, veluti exponentes, a laeva elementa ſingula primae permutationum a numeris 1, 2, 3 .... *r* oriundarum (prodit ${}^{1}a\,{}^{2}b\,{}^{3}c \ldots {}^{r}r$), et literis complexionis ſecundae elementa ſecundae permutationis, literis complexionis tertiae elementa tertiae permutationis, etc.

3) I. Complexionibus columnae (2), quae locis imparibus conſtitutae ſunt, praefigantur viciſſim ſigna + et — donec ${}^{1}a$ mutetur in ${}^{2}a$, deinde ſigna — et + donec ${}^{2}a$ mutetur in ${}^{3}a$, deinde iterum ſigna + et — donec ${}^{3}a$ mutetur in ${}^{4}a$, etc.
   II. Complexionum paribus locis conſtitutarum cuique praefigatur contrarium ſignum complexionis proxime praecedentis.

4) Repetantur ea quae ſupra (1, 2, 3) praecepta fuerunt, ſed tamen ita, ut in hac altera, quae prodit, columna ubique ponatur *s* loco *a*.

5) *Summa* complexionum cum ſuis ſignis columnae *poſterioris*, per *ſummam* prioris diviſa, valorem aequabit quantitatis α.

Pars

*) Hic literae *a, b, c,* .... ſignificant *coefficientes*, primum, ſecundum, tertium etc. literarum autem exponentes a laeva 1, 2, 3 .... *aequationes*, primam, ſecundam, tertiam etc. Sic verbi cauſa ${}^{2}c$ ſecundae aequationis tertium coefficientem denotat, et ${}^{3}a$ aequationis tertiae coefficientem primum.

Digitized by Google

Pars poſterior. Eodem modo inveniuntur $\beta, \gamma, \delta \ldots \varrho$, niſi quod s loco $b, c, d \ldots r$, reſpective ponatur (ut in 4. *Partis prioris* s pro a.).

Exempla.

1) Sit $r = 1$ et $\varrho = \alpha$. Ergo ${}^{1}s = {}^{1}a\alpha$ et

$$\alpha = {}^{1}s : {}^{1}a$$

2) Sit $r = 2$ et $\varrho = \beta$. Ergo $\begin{matrix} {}^{1}s = {}^{1}a\alpha + {}^{1}b\beta \\ {}^{2}s = {}^{2}a\alpha + {}^{2}b\beta \end{matrix}$ et

$$\alpha = \begin{bmatrix} + {}^{1}s{}^{2}b \\ - {}^{2}s{}^{1}b \end{bmatrix} : \begin{bmatrix} + {}^{1}a{}^{2}b \\ - {}^{2}a{}^{1}b \end{bmatrix}$$

$$\beta = \begin{bmatrix} + {}^{1}a{}^{2}s \\ - {}^{2}a{}^{1}s \end{bmatrix} : \begin{bmatrix} + {}^{1}a{}^{2}b \\ - {}^{2}a{}^{1}b \end{bmatrix}$$

3) Sit $r = 3$ et $\varrho = \gamma$. Ergo $\begin{matrix} {}^{1}s = {}^{1}a\alpha + {}^{1}b\beta + {}^{1}c\gamma \\ {}^{2}s = {}^{2}a\alpha + {}^{2}b\beta + {}^{2}c\gamma \\ {}^{3}s = {}^{3}a\alpha + {}^{3}b\beta + {}^{3}c\gamma \end{matrix}$ et

$$\alpha = \left\{\begin{matrix} + {}^{1}s{}^{2}b{}^{3}c \\ - {}^{1}s{}^{3}b{}^{2}c \\ - {}^{2}s{}^{1}b{}^{3}c \\ + {}^{2}s{}^{3}b{}^{1}c \\ + {}^{3}s{}^{1}b{}^{2}c \\ - {}^{3}s{}^{2}b{}^{1}c \end{matrix}\right\} : \left\{\begin{matrix} + {}^{1}a{}^{2}b{}^{3}c \\ - {}^{1}a{}^{3}b{}^{2}c \\ - {}^{2}a{}^{1}b{}^{3}c \\ + {}^{2}a{}^{3}b{}^{1}c \\ + {}^{3}a{}^{1}b{}^{2}c \\ - {}^{3}a{}^{2}b{}^{1}c \end{matrix}\right\} \quad \beta = \left\{\begin{matrix} + {}^{1}a{}^{2}s{}^{3}c \\ - {}^{1}a{}^{3}s{}^{2}c \\ - {}^{2}a{}^{1}s{}^{3}c \\ + {}^{2}a{}^{3}s{}^{1}c \\ + {}^{3}a{}^{1}s{}^{2}c \\ - {}^{3}a{}^{2}s{}^{1}c \end{matrix}\right\} : \left\{\begin{matrix} + {}^{1}a{}^{2}b{}^{3}c \\ - {}^{1}a{}^{3}b{}^{2}c \\ - {}^{2}a{}^{1}b{}^{3}c \\ + {}^{2}a{}^{3}b{}^{1}c \\ + {}^{3}a{}^{1}b{}^{2}c \\ - {}^{3}a{}^{2}b{}^{1}c \end{matrix}\right\} \quad \gamma = \left\{\begin{matrix} + {}^{1}a{}^{2}b{}^{3}s \\ - {}^{1}a{}^{3}b{}^{2}s \\ - {}^{2}a{}^{1}b{}^{3}s \\ + {}^{2}a{}^{3}b{}^{1}s \\ + {}^{3}a{}^{1}b{}^{2}s \\ - {}^{3}a{}^{2}b{}^{1}s \end{matrix}\right\} : \left\{\begin{matrix} + {}^{1}a{}^{2}b{}^{3}c \\ - {}^{1}a{}^{3}b{}^{2}c \\ - {}^{2}a{}^{1}b{}^{3}c \\ + {}^{2}c{}^{3}b{}^{1}c \\ + {}^{3}a{}^{1}b{}^{2}c \\ - {}^{3}a{}^{2}b{}^{1}c \end{matrix}\right\}$$

Severa ſolutionis hujus demonſtratio nititur in theoria *Variationum, omiſſis quidem repetitionibus*, id quod, alia ſcribendi occaſione data, oſtendam.

Scholion. Problematis ſolutio convenit cum regulis *Crameri* (*Introduction à l'Analyſe des Courbes* p. 656. ſeqq. *Num. I. et II. De l'Evanouiſſement des Inconnues*). Easdem HINDENBURGIUS, ſimul cum regulis de eadem re a BEZOUTO traditis (BEZOUT *Théorie générale des Equations algébriques*), ampliſſime pertractavit in praefatione libelli *Rüdigeriani*, cui titulus eſt: *Specimen analyticum de lineis curvis ſecundi ordinis*, ubi etiam p. XLVI — XLVIII. regula traditur datae complexionis permutationes reperiundi, qua h. l. (Schol. Pars prior 1, 2.) opus eſt.

§. XXXI.

PROBLEMA.

Propoſitae aequationis:

$$[1-(ax+bx^2\ldots)]^{\alpha}[1-(\mathfrak{a}x+\mathfrak{b}x^2\ldots)]^{\beta}\ldots[1-(\mathfrak{A}x+\mathfrak{B}x^2)]^{\varrho} = 1-(\mathrm{A}x+\mathrm{B}x^2+\mathrm{C}x^3\ldots+\mathrm{N}x^n\ldots)$$

exponentes incognitos $\alpha, \beta, \ldots \varrho$ *per coefficientes datos*, $a, b, c \ldots$; $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \ldots$; $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \ldots$; et $\mathrm{A}, \mathrm{B}, \mathrm{C} \ldots$ exprimere.

SOLU-

Digitized by Google

SOLUTIO.

Secundum §. XXVIII. poſito $n = 1, 2, 3, \ldots r$, eſt:

$$\underset{(A)}{i'J} = \underset{(a)}{i'J}\alpha + \underset{(\mathrm{a})}{i'J}\beta + \cdots\cdots + \underset{(\mathfrak{A})}{i'J}\varrho$$

$$\underset{(A,B)}{i'J} = \underset{(a,b)}{i'J}\alpha + \underset{(\mathrm{a},\mathrm{b})}{i'J}\beta + \cdots\cdots + \underset{(\mathfrak{A},\mathfrak{B})}{i'J}\varrho$$

$$\underset{(A,B,C)}{i'J} = \underset{(a,b,c)}{i'J}\alpha + \underset{(\mathrm{a},\mathrm{b},\mathrm{c})}{i'J}\beta + \cdots\cdots + \underset{(\mathfrak{A},\mathfrak{B},\mathfrak{C})}{i'J}\varrho$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$\underset{(A,B,\ldots R)}{i'J} = \underset{(a,b,\ldots r)}{i'J}\alpha + \underset{(\mathrm{a},\mathrm{b},\ldots \mathrm{r})}{i'J}\beta + \cdots\cdots + \underset{(\mathfrak{A},\mathfrak{B},\ldots \mathfrak{R})}{i'J}\varrho$$

Ex his aequationibus eruuntur (§. XXX.) valores literarum $\alpha.\beta\ldots\varrho$ quaeſiti. Nam,

1) poſito $r = 1$ et $\varrho = \alpha$, fit $\underset{(A)}{i'J} = \underset{(a)}{i'J}\alpha$. Ergo: $\alpha = + \underset{(A)}{i'J} : \underset{(a)}{i'J}$

2) poſito $r = 2$ et $\varrho = \beta$, fit $\begin{matrix} \underset{(A)}{i'J} = \underset{(a)}{i'J}\alpha + \underset{(\mathrm{a})}{i'J}\beta \\ \underset{(A,B)}{i'J} = \underset{(a,b)}{i'J}\alpha + \underset{(\mathrm{a},\mathrm{b})}{i'J}\beta \end{matrix}$ Ergo:

$$\alpha = \underset{(A,B)\,(\mathrm{a},\mathrm{b})}{\left\{\begin{matrix} +i'J & i'J \\ -i'J & i'J \end{matrix}\right\}} : \underset{(a,b)\,(\mathrm{a},\mathrm{b})}{\left\{\begin{matrix} +i'J & i'J \\ -i'J & i'J \end{matrix}\right\}} \qquad \beta = \underset{(a,b)\,(A,B)}{\left\{\begin{matrix} +i'J & i'J \\ -i'J & i'J \end{matrix}\right\}} : \underset{(a,b)\,(\mathrm{a},\mathrm{b})}{\left\{\begin{matrix} +i'J & i'J \\ -i'J & i'J \end{matrix}\right\}}$$

ubi quilibet index pertinet ad ſigna involutoria ſeriei verticalis ſuper ipſum poſitae.

Scholion. Dignum notari hoc problema, quippe methodorum vulgarium nulla, quod equidem ſciam, ad illud ſolvendum ſufficiat, ejusdem potius ſolutio videatur impoſſibilis, cum incognitae *plures* α, β, γ, ... ϱ, *una* ex aequatione ſint determinandae. *Analyſis* igitur *Combinatoria* h. l. uti ſaepius, aliis methodis efficit majora.

Sed haec jam ſufficiant, plura in additamentis.

ADDI-

Digitized by Google

ADDITAMENTA.

§ 1.

Summam libelli hactenus propoſita complectuntur. Varia Corollaria et Exempla, quae adhuc reliqua ſunt, nunc demum ſequuntur, quod relationibus §. XXVIII. – XXXI. exhibitis interjecta, ipſarum ordinem turbaſſent.

§ 2.

Si in aequatione §. XXVII. expoſita factorum multiplicandorum omnes ſunt binomiorum potentiae *) veluti:

$$(1-ax)^{\alpha}(1-bx)^{\beta}(1-cx)^{\gamma}(1-dx)^{\delta}\ldots(1-rx)^{\varrho}=1-(\mathrm{A}x+\mathrm{B}x^2+\mathrm{C}x^3+\mathrm{D}x^4+\ldots+\mathrm{N}x^n+\ldots)$$

tunc eſt (§. XXV.):

$$^{n}\Sigma=a^{n}\alpha+b^{n}\beta+c^{n}\gamma+d^{n}\delta\ldots+r^{n}\varrho$$

et ſequitur ex §. XXVIII. (mutato ſcilicet valore ſigni $^{n}\Sigma$)

$$I.\quad {}^{n}\Sigma=\underset{(\mathrm{A},\mathrm{B},\mathrm{C},\ldots\mathrm{N})}{{}^{n}\mathrm{J}}=a^{n}\alpha+b^{n}\beta+c^{n}\gamma+d^{n}\delta+\ldots+r^{n}\varrho$$

$$II.\quad {}^{n}\Sigma={}^{n-1}\Sigma\mathrm{A}+{}^{n-2}\Sigma\mathrm{B}+{}^{n-3}\Sigma\mathrm{C}+\ldots+{}^{n-m}\Sigma\mathrm{M}\ldots+{}^{1}\Sigma\overset{-1}{\mathrm{N}}+n\mathrm{N}=a^{n}\alpha+b^{n}\beta+c^{n}\gamma\ldots+r^{n}\varrho$$

atque ex §. XXIX.

$$III.\quad \mathrm{N}=\frac{1}{1}\mathfrak{a}^{n}\mathfrak{A}-\frac{1}{1.2}\mathfrak{b}^{n}\mathfrak{B}+\frac{1}{1.2.3}\mathfrak{c}^{n}\mathfrak{C}-\frac{1}{1.2.3.4}\mathfrak{d}^{n}\mathfrak{D}+\ldots\pm\frac{1}{1.2\ldots m}\mathfrak{m}^{n}\mathfrak{M}\mp\ldots\pm\frac{1}{1.2\ldots n}\mathfrak{n}^{n}\mathfrak{N}$$

$$\left(\begin{matrix}\frac{^{1}\Sigma}{1}, & \frac{^{2}\Sigma}{2}, & \frac{^{3}\Sigma}{3}, & \frac{^{4}\Sigma}{4}, & \ldots & \frac{^{n}\Sigma}{n}\end{matrix}\right)$$
$$\begin{matrix}1, & 2, & 3, & 4, & \ldots & n\end{matrix}$$

§. 3.

*) Hanc aequationem tractarunt T. Simpson (*Philoſophical Tranſactions Vol. XLVII. 1751. p. 20.*) et G. F. Tempelhofius, *Vir Strenuiſſimus*, (*Anfangsgründe der Analyſis des Unendlichen* p. 301.)

§. 3.

PROBLEMA.

Summam reperire ſeriei infinitae:

$$1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\frac{1}{5^{2n}}+\frac{1}{6^{2n}}+\frac{1}{7^{2n}}+\dots\dots,$$

in qua n eſſe poteſt numerus quilibet integer poſitivus.

SOLUTIO.

$$\frac{\mathit{Sin}\,z}{z}=\left(1-\frac{1.z^2}{1^2\pi^2}\right)\left(1-\frac{1.z^2}{2^2\pi^2}\right)\left(1-\frac{1.z^2}{3^2\pi^2}\right)\dots=1-\left(\frac{z^2}{1.2.3}-\frac{z^4}{1.2.3.4.5}+\dots\pm\frac{z^{2n}}{1.2\dots(2n+1)}\mp\dots\right)$$

Itaque ſecundum §. 2. formulam I *)

$${}^{n}\Sigma=\frac{(^{n})J}{\left(\frac{1}{1.2.3},\ \frac{-1}{1.2.3.4.5},\dots\frac{\mp 1}{1.2..(2n+1)}\right)}=\left(\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\dots\right)\frac{1}{\pi^{2n}}$$

et ſecundum formulam II.

$${}^{n}\Sigma=\frac{{}^{n-1}\Sigma}{1.2.3}-\frac{{}^{n-2}\Sigma}{1.2.3.4.5}+\frac{{}^{n-3}\Sigma}{1.2.3.4.5.6.7}-\dots\pm\frac{{}^{n-m}\Sigma}{1.2\dots(2m-1)}\mp\dots\mp\frac{{}^{1}\Sigma}{1.2..(2n-1)}\pm\frac{n}{1.2..(2n+1)}$$

$$=\left(\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\dots\right)\frac{1}{\pi^{2n}}$$

COROLLARIUM.

Multiplicetur per $\frac{2}{2^{2n}}$ aequatio ſequens:

1 +

*) Valores literarum in §. 2. hoc loco ſunt:

$x=z^2;\quad \alpha=1,\quad \beta=1;\quad \gamma=1;\quad \delta=1;\dots.$

$a=\frac{1}{1^2\pi^2};\ b=\frac{1}{2^2\pi^2};\ c=\frac{1}{3^2\pi^2};\ d=\frac{1}{4^2\pi^2};\dots.$

$A=\frac{1}{1.2.3};\quad B=\frac{-1}{1.2.3.4.5};\quad C=\frac{1}{1.2.3.4.5.6.7};\quad D=\frac{-1}{1.2.3.4.5.6.7.8.9};\dots.$

Digitized by Google

$$1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\frac{1}{5^{2n}}+\cdots\cdots = \frac{i^n J \pi^{2n}}{\left(\frac{1}{1.2.3},\ \frac{-1}{1.2.3.4.5},\ldots\frac{\pm 1}{1.2\ldots(2n+1)}\right)}$$

hinc oritur:

$$\frac{2}{2^{2n}}+\frac{2}{4^{2n}}+\frac{2}{6^{2n}}+\frac{2}{8^{2n}}+\cdots\cdots = \frac{\frac{2}{2^{2n}}\, i^n J \pi^{2n}}{\left(\frac{1}{1.2.3},\ \frac{-1}{1.2.3.4.5},\ldots\frac{\pm 1}{1.2\ldots(2n+1)}\right)}$$

qua ſerie de priori detracta, provenit:

$$1-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\frac{1}{5^{2n}}-\cdots\cdots = \frac{\left(1-\frac{1}{2^{2n-1}}\right) i^n J \pi^{2n}}{\left(\frac{1}{1.2.3},\ \frac{-1}{1.2.3.4.5}\ldots\frac{\pm 1}{1.2\ldots(2n+1)}\right)}$$

§. 4

Problema.

Summam reperire ſeriei infinitae:

$$1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\frac{1}{9^{2n}}+\frac{1}{11^{2n}}+\frac{1}{13^{2n}}+\cdots\cdots$$

in qua n numerus eſſe poteſt quilibet integer poſitivus.

Solutio.

$$\mathit{Cos}\, z=\left(1-\frac{2^2z^2}{1^2\pi^2}\right)\left(1-\frac{2^2z^2}{3^2\pi^2}\right)\left(1-\frac{2^2z^2}{5^2\pi^2}\right)\ldots=1-\left(\frac{z^2}{1.2}-\frac{z^4}{1.2.3.4}+\frac{z^6}{1.2.3.4.5.6}-\ldots\right)$$

Itaque ſecundum §. 2. formulam *I.* *)

*) Valores literarum in §. 2. hoc loco ſunt: $x = z^2$; porro $\alpha = 1$

$$ {}^{n}\Sigma = \frac{(^{n}J)}{\left(\frac{1}{1.2}, \frac{-1}{1.2.3.4}, \ldots \frac{\pm 1}{1.2\ldots 2n}\right)} = \left(1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \ldots\ldots\right) \frac{2^{2n}}{\pi^{2n}} $$

et ſecundum formulam II.

$$ {}^{n}\Sigma = \frac{{}^{n-1}\Sigma}{1.2} - \frac{{}^{n-2}\Sigma}{1.2.3.4} + \frac{{}^{n-3}\Sigma}{1.2.3.4.5.6} - \ldots \mp \frac{{}^{1}\Sigma}{1.2..(2n-2)} \pm \frac{n}{1.2\,..\,2n} = \left(1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} \ldots\right) \frac{2^{2n}}{\pi^{2n}} $$

§. 5.

Problema.

Invenire ſummam ſeriei infinitae

$$ \frac{1}{1^{\varkappa}} \mp \frac{1}{3^{\varkappa}} + \frac{1}{5^{\varkappa}} \mp \frac{1}{7^{\varkappa}} + \frac{1}{9^{\varkappa}} \mp \frac{1}{11^{\varkappa}} + \frac{1}{13^{\varkappa}} \mp \ldots\ldots, $$

ubi ſuperiora ſigna valent, ſi $\varkappa$ eſt numerus integer poſitivus impar, inferiora, ſi eſt par.

Solutio.

Secundum Eulerum (*Introductio in Analyſin infinitorum Lib. I.* §. 171. et 175) eſt, poſito $\frac{\pi x}{4} = z$:

$$ Cos\,z + Sin\,z \begin{cases} = \left(1 + \frac{4z}{1\pi}\right)\left(1 - \frac{4z}{3\pi}\right)\left(1 + \frac{4z}{5\pi}\right)\left(1 - \frac{4z}{7\pi}\right)\left(1 + \frac{4z}{9\pi}\right)\left(1 - \frac{4z}{11\pi}\right)\ldots \\ = 1 - \left(-\frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} - \frac{z^4}{1.2.3.4} - \ldots \mp \frac{z^{2n-1}}{1.2(2n-1)} \pm \frac{z^{2n}}{1.2\ldots 2n} \pm \ldots\right) \end{cases} $$

Itaque

$$ \alpha = 1; \quad \beta = 1; \quad \gamma = 1; \quad \delta = 1; \ldots. $$
$$ a = \frac{2^2}{2^2\pi^2};\ b = \frac{2^2}{3^2\pi^2};\ c = \frac{2^2}{5^2\pi^2};\ d = \frac{2^2}{7^2\pi^2}; \ldots. $$
$$ A = \frac{1}{1.2};\ B = \frac{-1}{1.2.3.4};\ C = \frac{1}{1.2.3.4.5.6};\ D = \frac{1-}{1.2.3.4.5.6.7.8}; \ldots. $$

Digitized by Google

Itaque ſecundum §. 2. formulam I. *)

$$-{}^{2n-1}\Sigma = \frac{-\,{}^{[2n-1]}J}{\left(\frac{-1}{1},\frac{1}{1.2},\frac{1}{1.2.3},\frac{-1}{1.2.3.4},\dots\frac{\mp 1}{1.2..(2n-1)}\right)} = \left(\frac{1}{1^{2n-1}}-\frac{1}{3^{2n-1}}+\frac{1}{5^{2n-1}}-\frac{1}{7^{2n-1}}+\dots\right)\frac{4^{2n-1}}{\pi^{2n-1}}$$

$$+{}^{2n}\Sigma = \frac{+\,{}^{[2n]}J}{\left(\frac{-1}{1},\frac{1}{1.2},\frac{1}{1.2.3},\frac{-1}{1.2.3.4},\dots\frac{\pm 1}{1.2.\,.2n}\right)} = \left(\frac{1}{1^{2n}}+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\dots\right)\frac{4^{2n}}{\pi^{2n}}$$

et ſecundum formulam II.

$$-{}^{2n-1}\Sigma = -\frac{{}^{2n-2}\Sigma}{1}+\frac{{}^{2n-3}\Sigma}{1.2}+\frac{{}^{2n-4}\Sigma}{1.2.3}-\dots\pm\frac{{}^{2}\Sigma}{1.1..(2n-3)} = \left(\frac{1}{1^{2n-1}}-\frac{1}{3^{2n-1}}+\frac{1}{5^{2n-1}}-\dots\right)\frac{4^{2n-1}}{\pi^{2n-1}}$$

$$+{}^{2n}\Sigma = -\frac{{}^{2n-1}\Sigma}{1}+\frac{{}^{2n-2}\Sigma}{1.2}+\dots\pm\frac{{}^{3}\Sigma}{1.2..(2n-3)}\mp\frac{{}^{2}\Sigma}{1.2..(2n-2)}\mp\frac{{}^{1}\Sigma}{1.2..(2n-1)} = \left(\frac{1}{1^{2n}}+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}\dots\right)\frac{4^{2n}}{\pi^{2n}}$$

in quibus formulis ſigna ſuperiora valent, ſi n eſt numerus impar, ſi par, inferiora.

Corollarium.

Quia (§. 3.)

$$1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\frac{1}{5^{2n}}+\dots = \frac{{}^{[n]}J\,\pi^{2n}}{\left(\frac{1}{1.2.3},\frac{-1}{1.2.3.4.5},\frac{1}{1.2.3.4.5.6.7},\dots\right)}$$

et

$$\frac{1}{2^{2n}}+\frac{1}{4^{2n}}+\frac{1}{6^{2n}}+\frac{1}{8^{2n}}+\dots = \frac{\frac{1}{2^{2n}}\,{}^{[n]}J\,\pi^{2n}}{\left(\frac{1}{1.2.3},\frac{-1}{1.2.3.4.5},\frac{1}{1.2.3.4.5.6.7},\dots\right)}$$

prodit,

*) Pro hac ſcilicet aequatione in §. 2.

$x = z;\ \alpha = 1;\ \beta = 1;\ \gamma = 1;\ \delta = 1;\dots$

$$a = -\frac{4}{1\pi};\ b = \frac{4}{3\pi};\ c = -\frac{4}{5\pi};\ d = \frac{4}{7\pi};\dots$$

$$A = \frac{-1}{1};\ B = \frac{1}{1.2};\ C = \frac{1}{1.2.3};\ D = \frac{-1}{1.2.3.4};\ E = \frac{-1}{1.2.3.4.5};\ F = \frac{1}{1.2.3.4.5.6};\dots$$

Digitized by Google

prodit, poſteriore ſerie de priore det. acta,

$$1+\frac{1}{3^{2m}}+\frac{1}{5^{2m}}+\frac{1}{7^{2m}}+\dots = \underset{\left(\frac{1}{1.2.3},\frac{-1}{1.2.3.4.5},\frac{1}{1.2.3\;4.5.6.7},\dots\right)}{\left(1-\frac{1}{2^{2m}}\right)[^{m}]J.\pi^{2m}}$$

Ejusdem ſeriei ſumma inventa eſt §. 4 et 5, unde ſequitur:

$$1+\frac{1}{3^{2m}}+\frac{1}{5^{2m}}\dots = \underset{\left(\frac{1}{1.2},\frac{-1}{1.2.3.4},\frac{1}{1.2.3\;4.5.6},\dots\right)}{[^{m}]J\left(\frac{\pi}{2}\right)^{2m}} = \underset{\left(\frac{1}{1.2.3},\frac{-1}{1.2.3.4.5},\frac{1}{1.2.\;7},\dots\right)}{\left(2^{2m}-1\right)[^{m}]J\left(\frac{\pi}{2}\right)^{2m}} = \underset{\left(\frac{-1}{1},\frac{+1}{1.2},\frac{+1}{1.2.3},\frac{-1}{1.2.3.4},\dots\right)}{\frac{1}{2^{2m}}[^{m}]J\left(\frac{\pi}{2}\right)^{2m}};$$

§. 6.

Serierum quarundam summae proponuntur independentes, quae terminorum recurrentium auxilio hucusque exhibitae sunt.

In egregio opere (*Verſuch einer neuen Summationsmethode. Berlin* 1788.) Pfaffius, *Vir Celeberrimus*, cum aliis, quas inſtituit, quaeſtionibus graviſſimis, magnum formularum recurrentium numerum tradidit, quibus multarum ſerierum ſummae ex circuli rectificatione pendentes exprimuntur. Ex iisdem nonnullae h. l. deliguntur, quarum ſummae arctiſſimo vinculo cum propoſitis (§. 3. 4. 5) conjunctae, earundem auxilio in formulas independentes transformantur.

Summam ſeriei infinitae, ubi *m* numerus integer poſitivus eſt,

$$1+\frac{1}{2^{2m}}+\frac{1}{3^{2m}}+\frac{1}{4^{2m}}+\frac{1}{5^{2m}}+\dots+\frac{1}{n^{2m}}+\dots$$

Pfaffius in opere laudato ita exhibet:

$$\frac{\pi^{2}}{1.2.3}\Sigma\frac{1}{n^{2m-2}}-\frac{\pi^{4}}{1.2.3.4.5}\Sigma\frac{1}{n^{2m-4}}+\dots\mp\frac{\pi^{2m-2}}{1.2\dots(2m-1)}\Sigma\frac{1}{n^{2}}\pm\frac{\pi^{2m}m}{1.2.3\dots(2m+1)}\;{}^{*)}$$

can-

*) Nempe ſigno Σ Pfaffius exprimit ſummam ſeriei infinitae, cujus terminus generalis ea eſt functio numeri *n*, quae huic ſigno adjicitur, ſic Σ N ſumma eſt omnium valorum functionis N, qui proveniunt, poſito $n = 1, 2, 3, 4 \dots$ Signo $\Sigma \pm N$ idem notat ſeriei ſummam, in qua terminorum ſigna (+ —) alternantur (Pfaff. ibid. p. 4. **).

eandem (§. 3) reperimus eſſe $i^{m}J\pi^{2m}$

$$\left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2\ldots7}, \ldots\right)$$

Pfaffiana formula recurrit, noſtra vero eſt independens. Haec praemittenda erant, ut ſequentia intelligantur, ubi Pf. cum numero adpoſito *Pfaffiani* operis paginam notat.

I. Summa ſeriei infinitae $Sin\,\varphi + \frac{Sin\,2\,\varphi}{2^{2m-1}} + \frac{Sin\,3\,\varphi}{3^{2m-1}} + \frac{Sin\,4\,\varphi}{4^{2m-1}} + \ldots\ldots$

$$= \varphi\,\Sigma\frac{1}{n^{2m-2}} - \frac{\varphi^3}{1.2.3}\Sigma\frac{1}{n^{2m-4}} + \cdots \mp \frac{\varphi^{2m-3}}{1.2\ldots(2m-3)}\Sigma\frac{1}{n^2} \pm \frac{\varphi^{2m-2}}{1.2\ldots(2m-2)}\frac{\pi}{2} \mp \frac{\varphi^{2m-1}}{1.2\ldots(2m-1)}\frac{1}{2} \qquad \text{(Pf. 10.}$$

$$= \varphi\, i^{m-1}J\pi^{2m-2} - \frac{\varphi^3}{1.2.3}i^{m-2}J\pi^{2m-4} + \cdots \mp \frac{\varphi^{2m-3}}{1.2\ldots(2m-3)}\,i^{1}J\pi^{2} \pm \frac{\varphi^{2m-2}}{1.2\ldots(2m-2)}\frac{\pi}{2} \mp \frac{\varphi^{2m-1}}{1.2\ldots(2m-1)}\frac{1}{2} \qquad \text{(§. 3.}$$

$$\left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2.3.4.5.6.7}, \frac{-1}{1.2.3\ldots9}, \ldots\right)$$

II. Summa ſeriei infinitae $Sin\,\varphi - \frac{Sin\,2\,\varphi}{2^{2m-1}} + \frac{Sin\,3\,\varphi}{3^{2m-1}} - \frac{Sin\,4\,\varphi}{4^{2m-1}} + \ldots$

$$= \varphi\,\Sigma\pm\frac{1}{n^{2m-2}} - \frac{\varphi^3}{1.2.3}\Sigma\pm\frac{1}{n^{2m-4}} + \cdots \mp \frac{\varphi^{2m-3}}{1.2\ldots(2m-3)}\Sigma\pm\frac{1}{n^2} \pm \frac{\varphi^{2m-1}}{1.2\ldots(2m-1)}\frac{1}{2} \qquad \text{(Pf. 8.}$$

$$= \varphi\left(1-\frac{1}{2^{2m-3}}\right)i^{m-1}J\pi^{2m-2} - \frac{\varphi^3}{1.2.3}\left(1-\frac{1}{2^{2m-5}}\right)i^{m-2}J\pi^{2m-4} \ldots \mp \frac{\varphi^{2m-3}}{1.2\ldots(2m-3)}\left(1-\frac{1}{2}\right)i^{1}J\pi^{2} \pm \frac{\varphi^{2m-1}}{1.2\ldots(2m-1)}\frac{1}{2} \qquad \text{(§.3.C.}$$

$$\left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2.3.4.5.6.7}, \ldots\ldots\right)$$

III. Summa ſeriei infinitae $Cos\,\varphi + \frac{Cos\,2\,\varphi}{2^{2m}} + \frac{Cos\,3\,\varphi}{3^{2m}} + \frac{Cos\,4\,\varphi}{4^{2m}} + \ldots\ldots$

$$= \Sigma\frac{1}{n^{2m}} - \frac{\varphi^2}{1.2}\,\Sigma\frac{1}{n^{2m-2}} + \ldots \pm \frac{\varphi^{2m-2}}{1.2.(2m-2)}\,\Sigma\frac{1}{n^2} \mp \frac{\varphi^{2m-1}}{1.2\ldots(2m-1)}\,\frac{\pi}{2} \pm \frac{1}{2}\,\frac{\varphi^{2m}}{1.2\ldots2m} \qquad \text{(Pf. 12.}$$

$$= i^{m}J\pi^{2m} - \frac{\varphi^2}{1.2}i^{m-1}J\pi^{2m-2} + \ldots \pm \frac{\varphi^{2m-2}}{1.2\ldots(2m-2)}i^{1}J\pi^{2} \mp \frac{\varphi^{2m-1}}{1.2\ldots(2m-1)}\,\frac{\pi}{2} \pm \frac{1}{2}\,\frac{\varphi^{2m}}{1.2\ldots2m} \qquad \text{(§. 3.}$$

$$\left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2.3.4.5.6.7}, \ldots\right)$$

IV. Sum-

Digitized by Google

IV. Summa ſeriei infinitae $Cos\,\varphi - \frac{Cos\,2\,\varphi}{2^{2m}} + \frac{Cos\,3\,\varphi}{3^{2m}} - \frac{Cos\,4\,\varphi}{4^{2m}} + \ldots$

$$= \Sigma \pm \frac{1}{n^{2m}} - \frac{\varphi^2}{1.2}\,\Sigma \pm \frac{1}{n^{2m-2}} + \ldots \pm \frac{\varphi^{2m-2}}{1.2..(2m-2)}\,\Sigma \pm \frac{1}{n^2} \mp \frac{1}{2}\,\frac{\varphi^{2m}}{1.2..2m} \qquad \text{(Pf. 12.}$$

$$= \left(1 - \frac{1}{2^{2m-1}}\right) {}^{2m}J\pi^{2m} - \frac{\varphi^2}{1.2}\left(1 - \frac{1}{2^{2m-3}}\right) {}^{2m-2}J\pi^{2m-2} \ldots \pm \frac{\varphi^{2m-2}}{1.2..(2m-2)}\left(1 - \frac{1}{2}\right) {}^{2}J\pi^2 \mp \frac{1}{2}\,\frac{\varphi^{2m}}{1.2..2m} \quad (\S.3\ \text{C.}$$

$$\left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2.3.4.5.6.7}, \ldots\right)$$

§. 7.

P r o b l e m a.

Summam reperire ſeriei infinitae

$$Cos\,\varphi - \frac{Cos\,3\,\varphi}{3^{2m-1}} + \frac{Cos\,5\,\varphi}{5^{2m-1}} - \frac{Cos\,7\,\varphi}{7^{2m-1}} + \ldots\ldots$$

in qua m numerus eſt integer poſitivus.

S o l u t i o.

Pfaffiana methodo Coſinus arcuum φ, $3\,\varphi$, $5\,\varphi$, ... ſolvantur in ſeries infinitas, eae multiplicentur reſpective per 1; $-\frac{1}{3^{2m-1}}$; $+\frac{1}{5^{2m-1}}$; ... deinde factores numeratori et denominatori ſingulorum terminorum communes extinguantur, termini ipſi ſecundum potentias arcus φ disponantur, et coefficientes dignitatum arcus φ in ſummas colligantur: prodibit auxilio §. 5. ſumma ſeriei propoſitae

$$= -\,{}^{2m-1}J\frac{\pi^{2m-1}}{4^{2m-1}} + {}^{2m-3}J\frac{\pi^{2m-3}}{4^{2m-3}}\,\frac{\varphi^2}{1.2} - \ldots \mp {}^{3}J\frac{\pi^3}{4^3}\,\frac{\varphi^{2m-4}}{1.2..(2m-4)} \pm {}^{1}J\frac{\pi^1}{4^1}\,\frac{\varphi^{2m-2}}{1.2..(2m-2)}$$

$$\left(\frac{-1}{1}, \frac{1}{1.2}, \frac{1}{1.2.3}, \frac{-1}{1.2.3.4}, \frac{-1}{1.2.3.4.5}, \ldots\right)$$

(Conferatur Pfaffius l. c. X. 3. p. 39. ſeq.)

§. 8.

Digitized by Google

§. 8.

Problema.

Summam reperire ſeriei infinitae, in qua m numerus eſt integer poſitivus:

$$\mathrm{Sin}\,\varphi - \frac{\mathrm{Sin}\,3\,\varphi}{3^{2m}} + \frac{\mathrm{Sin}\,5\,\varphi}{5^{2m}} - \frac{\mathrm{Sin}\,7\,\varphi}{7^{2m}} + \dots$$

Solutio.

Sinus ſolvantur in ſeries infinitas, caeteraque agantur uti §. 7. prodit ſumma quaeſita:

$$= -\{^{2m-1}]\frac{\pi^{2m-1}}{4^{2m-1}}\varphi + \{^{2m-3}]\frac{\pi^{2m-3}}{4^{2m-3}}\frac{\varphi^3}{1.2.3} - \dots \pm \{^{3}]\frac{\pi^3}{4^3}\frac{\varphi^{2m-3}}{1.2..(2m-3)} \mp \{^{1}]\frac{\pi^1}{4^1}\frac{\varphi^{2m-1}}{1.2..(2m-1)}$$

$$\left(\frac{-1}{1}, \frac{1}{1.2}, \frac{1}{1.2.3.}, \frac{-1}{1.2.3.4.}, \dots\right)$$

(Conferatur Pfaffius l. c. X. 5. p. 41.)

§. 9.

Eadem methodo, quae §. 7. et 8 adhibita eſt, ſed auxilio Corollarii §. 3. eruuntur ſummae ſerierum ſequentium, in quibus *m* numerus eſt integer poſitivus:

$$1)\ \frac{\mathrm{Sin}\,2\,\varphi}{2^{2m+1}} - \frac{\mathrm{Sin}\,4\,\varphi}{4^{2m+1}} + \frac{\mathrm{Sin}\,6\,\varphi}{6^{2m+1}} - \dots = \left(1 - \frac{1}{2^{2m-1}}\right)\{^{2m}]\frac{\pi^{2m}}{2^{2m}}\varphi - \left(1 - \frac{1}{2^{2m-3}}\right)\{^{2m-2}]\frac{\pi^{2m-2}}{2^{2m-2}}\frac{\varphi^3}{1.2.3} + \dots$$

$$\dots \pm \left(1 - \frac{1}{2}\right)\{^{2}]\frac{\pi^2}{2^2}\frac{\varphi^{2m-1}}{1.2..(2m-1)} \mp \frac{1}{2}\frac{\varphi^{2m+1}}{1.2..(2m+1)}$$

$$2)\ \frac{\mathrm{Cos}\,2\,\varphi}{2^{2m}} - \frac{\mathrm{Cos}\,4\,\varphi}{4^{2m}} + \frac{\mathrm{Cos}\,6\,\varphi}{6^{2m}} - \dots = \left(1 - \frac{1}{2^{2m-1}}\right)\{^{2m}]\frac{\pi^{2m}}{2^{2m}} - \left(1 - \frac{1}{2^{2m-3}}\right)\{^{2m-2}]\frac{\pi^{2m-2}}{2^{2m-2}}\frac{\varphi^2}{1.2} + \dots$$

$$\dots \pm \left(1 - \frac{1}{2}\right)\{^{2}]\frac{\pi^2}{2^2}\frac{\varphi^{2m-2}}{1.2..(2m-2)} \mp \frac{\varphi^{2m}}{1.2..2m}$$

ubi ad indicem $\left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2.3.4.5.6.7}, \dots\right)$ ſigna involutoria referuntur. Eſt enim ſecundum Eulerum (*Inſtitt. Calc. Diff. P. II.* §. 185.)

$$1 - 2^{2m} + 3^{2m} - 4^{2m} + 5^{2m} - \dots = 0$$

Hoc itaque loco ſerierum trigonometricarum ſummis, ſublato omni terminorum recurſu, campus patet latiſſimus.

§. 10.

Digitized by Google

§. 10.

Problema.

Producti ex factoribus numero infinitis, ſecundum potentias variabilis x ordinati,

$$\left(1-\frac{x}{b}\right)^{a}\left(1-\frac{x}{b^{2}}\right)^{a}\left(1-\frac{x}{b^{3}}\right)^{a}\cdots\cdots = 1-(\text{A}x+\text{B}x^{2}+\text{C}x^{3}+\ldots+\text{N}x^{n}+\ldots)$$

invenire coefficientem quemlibet a prioribus independenter.

Solutio.

Eſt hoc loco ſecundum §. 2. et formulam III.

$$\frac{{}^{n}\Sigma}{n}=\frac{1}{n}\left(\frac{a}{b^{n}}+\frac{a}{b^{2n}}+\frac{a}{b^{3n}}+\frac{a}{b^{4n}}+\ldots\right)=\frac{a}{n(b^{n}-1)}$$

atque

$$\text{N}=\frac{1}{1}\mathfrak{a}^{n}\mathrm{A}-\frac{1}{1.2}\mathfrak{b}^{n}\mathrm{B}+\frac{1}{1.2.3}\mathfrak{c}^{n}\mathrm{C}-\ldots\pm\frac{1}{1.2\ldots m}\mathfrak{m}^{n}\mathcal{M}\ldots\pm\frac{1}{1.2..n}\mathfrak{n}^{n}\mathcal{N}$$

$$\left(\underset{1,}{\frac{a}{1(b^{1}-1)}},\underset{2,}{\frac{a}{2(b^{2}-1)}},\underset{3,}{\frac{a}{3(b^{3}-1)}}\ldots\ldots\underset{n}{\frac{a}{n(b^{n}-1)}}\right)$$

unde prodit, poſito $n=1,\ 2,\ 3,\ldots$

$$\text{A}=\frac{a}{b-1};\quad \text{B}=\frac{a}{2(b^{2}-1)}-\frac{a^{2}}{2(b-1)^{2}};\quad \text{C}=\frac{a}{3(b^{3}-1)}-\frac{a^{2}}{2(b-1)(b^{2}-1)}+\frac{a^{3}}{2.3.(b-1)^{3}}$$

§. 11.

Quodſi in aequatione §. 2. ſubſtituatur

$$\left\{\begin{matrix}\text{pro} & \alpha, & \beta, & \gamma, & \delta, & \varepsilon, & \zeta,\ldots\\ \text{reſpective} & +\alpha, & -\alpha, & +\beta, & -\beta, & +\gamma, & -\gamma,\ldots\end{matrix}\right\}\text{ et }\left\{\begin{matrix}\text{pro} & a, & b, & c, & d, & e, & f,\ldots\\ \text{reſpective} & +a, & -a, & -b, & +b, & +c, & -c,\ldots\end{matrix}\right\}\text{ oritur}$$

$$(1-ax)^{\alpha}(1+ax)^{-\alpha}(1+bx)^{+\beta}(1-bx)^{-\beta}(1-cx)^{+\gamma}(1+cx)^{-\gamma}\ldots=1-(\text{A}x+\text{B}x^{2}+\text{C}x^{3}\ldots+\text{N}x^{n}..)$$

ubi numerus factorum ſemper eſt par, ſive ſit infinitus, ſive finitus, atque fit:

$$^{2m-1}\Sigma=2\left[a^{2m-1}\alpha-b^{2m-1}\beta+c^{2m-1}\gamma-d^{2m-1}\delta+\ldots\right]$$
$$^{2m}\Sigma=0$$

Hine

Hinc provenit:

$$\text{I.}\quad {}^{2m-1}\Sigma = {}^{2m-1}\Sigma B + {}^{2m-3}\Sigma D + \dots + {}^{1}\Sigma \overset{m-2}{M} + (2m-1)\overset{m-1}{M} = 2\left[a^{2m-1}\alpha - b^{2m-1}\beta + c^{2m-1}\gamma - \dots\right]$$

$$\text{II.}\quad {}^{2m}\Sigma = {}^{2m-1}\Sigma A + {}^{2m-3}\Sigma C + \dots + {}^{3}\Sigma \overset{m-1}{M} + {}^{1}\Sigma \overset{m-1}{M} + 2m\overset{m}{M} = 0$$

$$\text{III.}\quad N = \pm\left(\frac{1}{1.2..n} n^n \mathcal{N} + \frac{1}{1.2..(n-2)} n^n \overset{-2}{\mathcal{N}} + \frac{1}{1.2..(n-4)} n^n \overset{-4}{\mathcal{N}} + \dots\dots + \left\{\begin{matrix}\frac{1}{1} a^n A \\ \frac{1}{1.2} b^n B\end{matrix}\right.\right) \text{ *)}$$

$$\left(\begin{matrix}\frac{{}^1\Sigma}{1}, & c, & \frac{{}^3\Sigma}{3}, & o, & \frac{{}^5\Sigma}{5}, & o, & \frac{{}^7\Sigma}{7}, & \dots\dots \\ 1, & 2, & 3, & 4, & 5, & 6, & 7, & \dots\dots\end{matrix}\right)$$

in qua tertia formula $\begin{cases}\text{ſuperiora ſigna } + \text{ et } \frac{1}{1}a^nA \text{ valent, ſi } n \text{ numerus eſt impar,} \\ \text{inferiora } \ldots - \text{ et } \frac{1}{1.2}b^nB, \text{ ſi } n \text{ eſt par.}\end{cases}$

§. 12.

Problema.

Invenire valores ſecundum potentias quantitatis x digeſtos productorum, quae ſequuntur:

$$1)\ \left(\frac{1-x}{1+x}\right)^{Sin\,\varphi}\left(\frac{1+\frac{1}{2}x}{1-\frac{1}{2}x}\right)^{Sin\,2\varphi}\left(\frac{1-\frac{1}{3}x}{1+\frac{1}{3}x}\right)^{Sin\,3\varphi}\left(\frac{1+\frac{1}{4}x}{1-\frac{1}{4}x}\right)^{Sin\,4\varphi}\dots = 1-(Ax+Bx^2+Cx^3\dots+Nx^n\dots)$$

$$2)\ \left(\frac{1-x}{1+x}\right)^{Cos\,\varphi}\left(\frac{1+\frac{1}{3}x}{1-\frac{1}{3}x}\right)^{Cos\,3\varphi}\left(\frac{1-\frac{1}{5}x}{1+\frac{1}{5}x}\right)^{Cos\,5\varphi}\left(\frac{1+\frac{1}{7}x}{1-\frac{1}{7}x}\right)^{Cos\,7\varphi}\dots = 1-(\mathfrak{A}x+\mathfrak{B}x^2+\mathfrak{C}x^3\dots+\mathfrak{N}x^n\dots)$$

Solutio.

Secundum §. 11. **) eſt in producto 1)

$$ {}^{2m-1}\Sigma = 2\left[Sin\ \varphi - \frac{Sin\ 2\ \varphi}{2^{2m-1}} + \frac{Sin\ 3\ \varphi}{3^{2m-1}} - \frac{Sin\ 4\ \varphi}{4^{2m-1}} + \dots\dots\right]$$

in

*) Scripſi h. l. $n^n\overset{-1}{\mathcal{N}}$ pro $\overset{-1}{n}{}^{n}\overset{-1}{\mathcal{N}}$ etc. quod hoc compendio nulla obſcuritas naſcitur.

**) Eſt ſcilicet in §. 11. intuitu prioris producti

$\alpha = Sin\ \varphi;\ \beta = Sin\ 2\ \varphi;\ \gamma = Sin\ 3\varphi;\dots\ a=1;\ b=\frac{1}{2};\ c=\frac{1}{3};\ d=\frac{1}{4};\dots$

intuitu poſterioris

$\alpha = Cos\ \varphi;\ \beta = Cos\ 3\ \varphi;\ \gamma = Cos\ 5\varphi;\dots\ a=1;\ b=\frac{1}{3};\ c=\frac{1}{5};\ d=\frac{1}{7};\dots$

Digitized by Google

in producto 2)

$${}^{2m-1}\Sigma = 2\left[Cos\,\varphi - \frac{Cos\,3\,\varphi}{3^{2m-1}} + \frac{Cos\,5\,\varphi}{5^{2m-1}} - \frac{Cos\,7\,\varphi}{7^{2m-1}} + \ldots\ldots\right]$$

cum vero harum ſerierum ſummae cognitae ſint (§. 6. II. et §. 7.), ſimul utriusque ſeriei coefficiens quilibet independenter erui poteſt. Invenitur:

$$\text{A} = \varphi; \quad \text{B} = -\frac{\varphi^2}{1.2}; \quad \text{C} = \frac{\varphi^3}{1.2.3}\left(\frac{2\,\varphi^2 + \pi^2}{3}\right); \ldots\ldots$$

$$\mathfrak{A} = \frac{\pi}{2}; \quad \mathfrak{B} = -\frac{\pi^2}{8}; \quad \mathfrak{C} = \frac{\pi}{4}\left(\frac{\pi^2}{4^2} + \frac{\varphi^2}{3}\right); \ldots\ldots$$

§. 13.

Ex aequatione

$$(1-ax)(1-bx)(1-cx)\ldots(1-rx) = 1 - (\text{A}x + \text{B}x^2 + \text{C}x^3 + \ldots + \text{N}x^n \ldots + \text{R}x^r)$$

ſequitur ſecundum §. 2.

$$\text{I.}\quad {}^n\Sigma = \underset{(\text{A, B, C, \ldots N})}{{}^n\text{J}} = a^n + b^n + c^n + \ldots + r^n$$

$$\text{II.}\quad {}^n\Sigma = {}^{n-1}\Sigma\text{A} + {}^{n-2}\Sigma\text{B} + \ldots + {}^{n-1}\Sigma\text{N} + n\text{N} = a^n + b^n + c^n + \ldots + r^n$$

$$\text{III.}\quad \text{N} = \frac{1}{1}\,\mathfrak{a}^n\text{A} - \frac{1}{1.2}\,\mathfrak{b}^n\text{B} + \frac{1}{1.2.3}\,\mathfrak{c}^n\text{C} - \ldots \pm \frac{1}{1.2..m}\,\mathfrak{m}^n\mathcal{M} \mp \ldots \pm \frac{1}{1.2..n}\,\mathfrak{n}^n\mathcal{N}$$

$$\left(\frac{{}^1\Sigma}{1}, \frac{{}^2\Sigma}{2}, \frac{{}^3\Sigma}{3}, \ldots \frac{{}^n\Sigma}{n}\right)$$

Quodſi ponatur $1 - (\text{A}x + \text{B}x^2 + \ldots + \text{R}x^r) = 0$, ſimul $a,\ b,\ c, \ldots r$; ſunt radices aequationis

$$y^r - (\text{A}y^{r-1} + \text{B}y^{r-2} + \text{C}y^{r-3} + \ldots + \text{N}y^{r-n} + \ldots + \text{R}) = 0$$

in qua $y = \frac{1}{x}$ (*) et formularum propoſitarum prima et ſecunda exhibent ſummam radicum aequa-

*) Ex aequatione $(1-ax)(1-bx)\ldots(1-rx) = 1-(\text{A}x + \text{B}x^2 \ldots + \text{R}x^r) = 0$ ſequitur,

poſito $x = \frac{1}{y}$; $(1-\frac{a}{y})(1-\frac{b}{y})\ldots(1-\frac{r}{y}) = 1-(\frac{\text{A}}{y} + \frac{\text{B}}{y^2} + \ldots + \frac{\text{R}}{y^r}) = 0$,

ſive $(\frac{y-a}{y})(\frac{y-b}{y})\ldots(\frac{y-r}{y}) = 1-(\frac{\text{A}}{y} + \frac{\text{B}}{y^2} + \ldots + \frac{\text{R}}{y^r}) = 0$

et ſi haec aequatio multiplicetur per y^r

$$(y-a)(y-b)\ldots(y-r) = y^r - [\text{A}y^{r-1} + \text{B}y^{r-2} \ldots + \text{R}\,y^0] = 0$$

Digitized by Google

aequationis hujus ad potentiam *n*tam elevatarum; altera quidem independenter, altera vero, Newtoni, Baermanni, Kaestneri, Euleri, Tempelhofii, aliorumque Analyſtarum exemplo *), inſertis praecedentibus radicum potentiis.

De tertia formula unum moneam: Exhibet ea valorem coefficientis N, ſed conſtat, eundem aequare *n*tam claſſem omnium complexionum rite ordinatarum indicis, (*a*, *b*, *c*, ... *r*), in quibus ſingulis nullum elementum bis vel ſaepius occurrit, (Kaestneri *Analyſis endl. Gröſsen* §. 224.) i. e. *n*tam claſſem *Combinationum ſimpliciter*, *indicis* (*a*, *b*, *c*, *d*, ... *r*) *et omiſſis quidem repetitionibus* **), quam Hindenburgius, *Vir Celeberrimus*, hoc notat ſigno:

$$\underset{(a,\ b,\ c,\ \ldots r)}{\mathcal{N}'}$$

Itaque, duplici valore coefficientis N invento, haec prodit relatio:

$$\mathrm{N} = \underset{(a,b,c,.\,r)}{{}^{n}\mathcal{N}'} = \frac{1}{1}a^{n}\mathrm{A} - \frac{1}{1{,}2}b^{n}\mathrm{B} + \ldots\ldots \pm \frac{1}{1.2..m}m^{n}\mathcal{M} \mp \ldots\ldots \pm \frac{1}{1.2..n}n^{n}\mathcal{N}$$

$$\left(\frac{{}^{1}\Sigma}{1},\ \frac{{}^{2}\Sigma}{2},\ \frac{{}^{3}\Sigma}{3},\ \ldots\ldots\ \frac{{}^{n}\Sigma}{n}\right)$$ §. 14.

*) Theorema, quod formula ${}^{n}\Sigma = {}^{n-1}\Sigma\mathrm{A} + {}^{n-2}\Sigma\mathrm{B} + \ldots\ldots + {}^{1}\Sigma\overset{n-1}{\mathrm{N}} + n\mathrm{N}$ exprimitur, Newtonus (*Arithmetica Univerſalis in fine capitis de transmutationibus aequationum p.* 192. *Editionis s' Graveſandinae*) propoſuit, ſed nullam ejus demonſtrationem. Kaestnerus, *Vir Illuſtris*, illud demonſtravit, afferens ſimul alia, quae pertinent ad hoc theorema (*Analyſis endlicher Gröſsen* §. 751). *Eulerianas* hujus theorematis demonſtrationes, Michelsen, *Vir Celeberrimus*, in additamentis ſuis ad Euleri *Introductionem in Analyſin infinitorum* collegit (*Zuſätze zum* 10*ten Capitel des* 1*ſten Buchs*).

**) In tabula adpoſita Combinationum ſimpliciter, indicis (*a*, *b*, *c*, *d*, *e*) et omiſſis quidem repetitionibus, duarum vicinarum claſſium poſterior ex priori oritur, ſi quaelibet prioris claſſis complexio ante indicis elementa, noviſſimum ipſius elementum inſequentia, ſucceſſive ponitur, atque ita complexiones ordinantur, ut quae in idem deſinunt elementum, eadem in ſerie verticali collocentur.

a	*b*	*c*	*d*	*e*)	= A'
	ab	*ac*	*ad*	*ae*	= B'
		bc	*bd*	*be*	
			cd	*ce*	
				de	
		abc	*abd*	*abe*	= C'
			acd	*ace*	
				ade	
			bcd	*bce*	
				bde	
				cde	
			abcd	*abce*	= D'
				abde	
				acde	
				bcde	
				abcde	= E' (*a*, *b*, *c*, *d*, *e*)

§. 14.

PROBLEMA.

Data aequatione

$$[1-(1x+1x^2)]^{\alpha}[1-(2x+2x^2)]^{\beta}=1-(3x+4x^2+\mathrm{C}x^3+\mathrm{D}x^4+\ldots)$$

reperire exponentes α, β *et coëfficientes* C, D, E, ...

SOLUTIO.

Pars prior. Eſt

$$\begin{array}{lll} \mathrm{i}^1\mathrm{J}=1 & \mathrm{i}^1\mathrm{J}=2 & \mathrm{i}^1\mathrm{J}=3 \\ \mathrm{i}^2\mathrm{J}=3 & \mathrm{i}^2\mathrm{J}=8 & \mathrm{i}^2\mathrm{J}=17 \\ \mathrm{i}^3\mathrm{J}=4 & \mathrm{i}^3\mathrm{J}=20 & \mathrm{i}^3\mathrm{J}=63+3\mathrm{C} \\ \mathrm{i}^4\mathrm{J}=7 & \mathrm{i}^4\mathrm{J}=56 & \mathrm{i}^4\mathrm{J}=257+12\mathrm{C}+4\mathrm{D} \\ (1,1) & (2,2) & (3,4,\mathrm{C},\mathrm{D}) \end{array}$$

Secundùm §. XXXI. reperitur:

$$\alpha=\underset{(3,4)\,(2,2)}{\begin{Bmatrix}+\mathrm{i}^1\mathrm{J} & \mathrm{i}^2\mathrm{J}\\ -\mathrm{i}^2\mathrm{J} & \mathrm{i}^1\mathrm{J}\end{Bmatrix}}:\underset{(1,1)\,(2,2)}{\begin{Bmatrix}+\mathrm{i}^1\mathrm{J} & \mathrm{i}^2\mathrm{J}\\ -\mathrm{i}^2\mathrm{J} & \mathrm{i}^1\mathrm{J}\end{Bmatrix}}=-5;\quad \beta=\underset{(1,1)\,(3,4)}{\begin{Bmatrix}+\mathrm{i}^1\mathrm{J} & \mathrm{i}^2\mathrm{J}\\ -\mathrm{i}^2\mathrm{J} & \mathrm{i}^1\mathrm{J}\end{Bmatrix}}:\underset{(1,1)\,(2,2)}{\begin{Bmatrix}+\mathrm{i}^1\mathrm{J} & \mathrm{i}^2\mathrm{J}\\ -\mathrm{i}^2\mathrm{J} & \mathrm{i}^1\mathrm{J}\end{Bmatrix}}=4$$

Pars poſterior. Secundum §. XXVIII.

$$\underset{(3,4,\mathrm{C})}{\mathrm{i}^3\mathrm{J}}=-\underset{(1,1)}{\mathrm{i}^3\mathrm{J}}5+\underset{(2,2)}{\mathrm{i}^3\mathrm{J}}4;\quad \text{hoc eſt:}\quad 63+3\mathrm{C}=-4.5+20.4$$

ex qua aequatione eruitur $\mathrm{C}=-1$. Secundum eundem §.

$$\underset{(3,4,\mathrm{C},\mathrm{D})}{\mathrm{i}^4\mathrm{J}}=-\underset{(1,1)}{\mathrm{i}^4\mathrm{J}}5+\underset{(2,2)}{\mathrm{i}^4\mathrm{J}}4\quad \text{hoc eſt:}\quad 257+12\mathrm{C}+4\mathrm{D}=-7.5+56.4$$

unde ſequitur: $\mathrm{D}=-14$.

Eadem ratione valor reliquorum coefficientium E, F, G... ſucceſſive invenitur.

His licet paucis exemplis relationum ſupra §. XXVIII. ſeqq. propoſitarum uſus apparebit uberrimus.

EX OFFICINA KRAMERIANA

Digitized by Google

Pag.	*Lin.*	*Errata.*	*Corrige*
1	4	adhibitae	adhibitae novitatem
3	12	(§. Def. 1)	(§. II. Def. 1)
4	4	113444	11344.
—	8	III.	II.
7	6 a fine	123...γ	1.2.3...γ
8	19	$5ccdd$	$6ccdd$
9	16	$\frac{}{1}$	$\frac{\gamma}{1}$
—	ult.	fit	unde fit
11	9	${}^{\gamma}\mathfrak{A}a^{\gamma-2}\alpha^{\prime}A$	${}^{\gamma}\mathfrak{A}a^{\gamma-1}\alpha^{\prime}A$
—	6 a fine	his	hic
13	9	${}^{\gamma}\mathfrak{A}a\ \alpha^{\prime}A$	${}^{\gamma}\mathfrak{A}a^{\gamma-1}\alpha^{\prime}A$
—	12	$p\varkappa^{(m+1)}$	$p^{\prime}\varkappa^{(m+1)}$
—	13 a fine	Haec	Hae
14	13	illa	illae
—	6 a fine	*Variationum* n	*Variationum ſummas* n
16	11	adponantur	adponatur
—	12	duabus	duobus
20	8	$\begin{pmatrix} a, b, c, d, \ldots n \\ 1, 2, 3, 4, \ldots n \end{pmatrix}$	$\begin{pmatrix} a, b, c, d, \ldots n, \ldots \\ 1, 2, 3, 4, \ldots n, \ldots \end{pmatrix}$
21	6 a fine	XV	XVI
23	6 a fine	$\frac{1}{1}\mathfrak{b}^{\prime}B$	$\frac{1}{2}\mathfrak{b}^{\prime}B$
24	ult.	$\left(\frac{{}^{1}\Sigma}{1}x+\frac{{}^{2}\Sigma}{2}x+\cdots+\frac{\Sigma}{n}x^{n}\ldots\right)$	$-\left(\frac{{}^{1}\Sigma}{1}x+\frac{{}^{2}\Sigma}{2}x^{2}+\cdots+\frac{{}^{n}\Sigma}{n}x^{n}\ldots\right)$
26	9 a fine	$\begin{pmatrix} A, B, C, D, \ldots N \\ 1, 2, 3, 4, \ldots n \end{pmatrix}$	$\begin{pmatrix} A, B, C, D, \ldots N, \ldots \\ 1, 2, 3, 4, \ldots n, \ldots \end{pmatrix}$
27	11	nn	n n
28	14	$+ + +$	$\pm \mp \pm$ (uti pag. 29. lin. 8.)
31	*Ex.* no. 3	$r = s$	$r = 3$
36	2 a fine	$\frac{2^{1}}{5^{2}\pi^{2}}\omega$	$\frac{2^{2}}{5^{2}\pi^{2}};d$

Digitized by Google

Pag.	*Lin.*	*Errata.*	*Corrige.*
37	5 et 7 a fine	$\frac{1}{1.2.3.4.5.6}$	$\frac{1}{1.2.3.4.5.6.7}$
38	5	nventa	inventa
—	10	EXHIBITA	EXHIBITAE
39	9	$\frac{Sin\,2\varphi}{2^{2m-1}}$	$\frac{Sin\,2\varphi}{2^{2m-1}}$
—	6	$\varphi\Sigma\frac{1}{2^{2m-2}}-\frac{\varphi^3}{1.2.3}\Sigma\frac{1}{2^{2m-4}}$	$\varphi\Sigma\frac{1}{n^{2m-2}}-\frac{\varphi^3}{1.2.3}\Sigma\frac{1}{n^{2m-4}}$
—	7	$+\frac{\varphi^{2m-1}}{1.2..(2m-1)}\frac{1}{2}$	$\mp\frac{\varphi^{2m-1}}{1.2..(2m-1)}\frac{1}{2}$
40	5 a fine	colligantur	colligantur:
42	4 a fine	$(-bx)^{-\beta}$	$(1-bx)^{-\beta}$
43	9 a fine	$\left(\frac{1-\frac{1}{3}x}{+\frac{1}{3}x}\right)^{Sin\,3\varphi}$	$\left(\frac{1-\frac{1}{3}x}{1+\frac{1}{3}x}\right)^{Sin\,3\varphi}$

C79827

Digitized by Google

$$p^{\nu} = (az$$

$$p^{\nu} = p^{\nu}\varkappa \; \ldots\ldots\ldots\ldots + p^{\nu}\varkappa(n+1)z^{\nu\mu+n\delta} + \ldots\ldots$$

$$p^{\nu} = a^{\nu} \; \ldots\ldots\ldots\ldots \left.\begin{array}{l} + {}^{\nu}\mathfrak{A}\; a^{\nu-1}\; \mathfrak{a}\; {}^{n}A \\ + {}^{\nu}\mathfrak{B}\; a^{\nu-2}\; \mathfrak{b}\; {}^{n}I \\ + {}^{\nu}\mathfrak{C}\; a^{\nu-3}\; \mathfrak{c}\; {}^{n}C \\ + {}^{\nu}\mathfrak{D}\; a^{\nu-4}\; \mathfrak{d}\; {}^{n}D \\ + {}^{\nu}\mathfrak{E}\; a^{\nu-5}\; \mathfrak{e}\; {}^{n}E \\ + {}^{\nu}\mathfrak{F}\; a^{\nu-6}\; \mathfrak{f}\; {}^{n}F \\ \quad\;\; '' \\ \quad\;\; '' \\ + {}^{\nu}\mathfrak{M}\; a^{\nu-m}\; \mathfrak{m}^{n}\mathcal{M} \\ \quad\;\; '' \\ \quad\;\; '' \\ + {}^{\nu}\mathfrak{N}\; a^{\nu-n}\; \mathfrak{n}^{n}\mathcal{N} \end{array}\right| z^{\nu\mu+n\delta} + \ldots\ldots$$

$$p^{\nu} = \left.\begin{array}{l} a\,h \\ 2bg \\ 2cf \\ 2de \\ \\ 3b^2f \\ 6bce \\ 3bd^2 \\ 3c^2d \end{array}\right| z^{\nu\mu+7\delta} + \left.\begin{array}{l} {}^{\nu}\mathfrak{A}a^{\nu-1}\; i \\ + {}^{\nu}\mathfrak{B}a^{\nu-2}\left\{\begin{array}{l} 2bh \\ 2cg \\ 2df \\ e^2 \end{array}\right. \\ + {}^{\nu}\mathfrak{C}a^{\nu-3}\left\{\begin{array}{l} 3b^2g \\ 6bcf \\ 6bde \\ 3c^2e \\ 3cd^2 \end{array}\right. \end{array}\right| z^{\nu\mu+8\delta} + \left.\begin{array}{l} {}^{\nu}\mathfrak{A}a \quad k \\ + {}^{\nu}\mathfrak{B}a^{\nu-2}\left\{\begin{array}{l} 2bi \\ 2ch \\ 2dg \\ 2ef \end{array}\right. \\ + {}^{\nu}\mathfrak{C}a^{\nu-3}\left\{\begin{array}{l} 3b^2h \\ 6bcg \\ 6bdf \\ 3be^2 \\ 3c^2f \\ 6cde \\ d^3 \end{array}\right. \end{array}\right| z^{\nu\mu+9\delta} + \ldots\ldots$$

Digitized by Google

BIBLIOTECA NAZIONALE NAPOLI

C79827

Digitized by Google

Digitized by Google

Digitized by Google

Digitized by Google

BIBLIOTECA
N